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Wolfgang Buchholz, Richard Cornes, Dirk Rübbelke

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By Wolfgang Buchholz, University of Regensburg and CESifo Munich
Richard Cornes, Australian National University
Dirk Rübbelke, Technische Universität Bergakademie Freiberg

Summary

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JEL Classification: C78, H41

First thoughts and preliminary drafts on applying the Kolm triangle originated from Richard Cornes. Sadly, Richard Cornes passed away in August 2015.

Address for correspondence:
Dirk Rübbelke
Technische Universität Bergakademie Freiberg
Akademiestraße 6
09599 Freiberg
Germany
E-mail: dirk.ruebbelke@vwl.tu-freiberg.de

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Wolfgang Buchholz
University of Regensburg and CESifo Munich, Germany
e-mail: wolfgang.buchholz@wiwi.uni-regensburg.de

Richard Cornes
F. H. Gruen Chair, Research School of Economics
Australian National University, Canberra, Australia

Dirk Rübbelke
Technische Universität Bergakademie Freiberg, Germany
e-mail: dirk.ruebbelke@vwl.tu-freiberg.de

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1. Introduction

The Kolm triangle (see [1], Chapter 9) is a well-known and frequently used graphical device to visualize public good allocations in an economy with two agents (see [2], [3] and [4]). The advantage of the Kolm triangle approach, as compared to most other graphical methods for representing allocations in a public good economy, is that the aggregate budget constraint, the levels of both agents' private consumption, and the level of public good supply directly show up in the same diagram.[1] The Kolm triangle method has been particularly helpful to describe the Nash equilibrium in the case of non-cooperative public good provision and to compare this outcome with Pareto efficient public good allocations. Furthermore, the Kolm triangle approach facilitates the analysis of mechanisms for attaining an efficient public good allocation like the Lindahl equilibrium (see [9], [10], [11], [12] and [13]) as well as the study of preconditions and limitations faced by such mechanisms (see [14]). In this vein, we will show in this paper, how the Kolm triangle approach can be applied to "matching", which is another widely discussed approach aiming at improving public good allocation.

Following the seminal work of [15], matching in a public good economy means that agents subsidize the other agents' direct ("flat") public good contributions, about which the agents decide non-cooperatively as in the standard case of voluntary public good supply. By reducing the effective personalized public good price of the "matched" agents, matching leads to a Nash equilibrium with higher public good supply (see [16], [17], [18], [19], [20], [21], [22], and [23]).

The objective of this paper is to show that the Kolm triangle method allows for a catchy and intuitive graphical illustration of important features of matching mechanisms, which to a certain extent have already been treated analytically in the literature. That the functioning of matching schemes is quite often analyzed for public good economies with only two agents (see [24], [8], [25] and [26]) fits well to applying the Kolm triangle approach for an analysis of matching mechanisms.

[1] The most common visualization of public good allocations in a two-person economy uses a diagram with a horizontal and a vertical axis, in which the public good contributions of two agents are plotted at the two axis (see, e.g., [5], p. 154) but in which the level of public good supply does not show up directly. Alternative approaches are the use of a triangular Edgeworth box (e.g. [6], or [7]), which has some similarities with the Kolm triangle, or the use of a double diagram with public good supply at the vertical axis and private consumption of two agents at the horizontal axis going to the right and to the left, respectively (e.g. [8]). In this visualization, however, the budget constraint only appears indirectly.

We will proceed as follows: After describing the Kolm triangle approach in Section 2, we apply it in Section 3 to depict the changes in private consumption and public good supply that are implied by matching when both agents make strictly positive flat contributions to the public good. Yet such interior matching equilibria can only be expected for specific distributions of income between the two agents as has been shown in [27]. In Section 4, the range of income distributions, for which interiority prevails, is described in the Kolm triangle for the special case in which both agents have the same preferences. In Section 5, we then deal with corner matching equilibria especially focusing on the participation constraint, i.e. on the requirement that both agents are made better off through matching as compared to the original Nash equilibrium without matching. The fulfillment of the participation constraint, which sets limits to the application of a matching mechanism, also depends on the income distribution. For the description of this dependency (and its comparison with the conditions for interiority of the matching equilibrium) the Kolm triangle also turns out to be particularly useful and helps to present a novel contribution to the matching literature.

2. The Kolm triangle approach

We assume that there are two agents $i=1,2$ with utility functions $u^{(i)}(x_i,G)$ where x_i denotes agent i's private good consumption and G is public good supply. Both utility functions have the standard properties, i.e. they are twice partially differentiable and strictly monotone increasing in both variables and quasi-concave. Moreover, the private and the public good are assumed to be non-inferior for agent $i=1,2$. The public good is produced by a summation technology, for which we assume that the marginal rate of transformation between the private and the public good (and hence the technically given public good price) is equal to one. If w_i denotes agent i's initial endowment ("income") as measured in units of the private good an allocation (x_1,x_2,G) thus is feasible given the aggregate endowment $W=w_1+w_2$ (and no resources are wasted) if and only if $x_1+x_2+G=W$.

In this economy a matching mechanism is described by the two matching rates μ_1 and μ_2. The matching rate $\mu_i \geq 0$ indicates by how much agent i subsidizes agent j's flat contribution to the public good ($i,j=1,2$), i.e. how much agent i has committed to add to each unit of agent j's direct public good contribution. Agent i's total public good contribution then consists of two parts: Her direct flat contribution $z_i \geq 0$, on which she decides herself, and her indirect matching contribution $\mu_i z_j$, which is determined by the direct public good contribution z_j chosen by the other agent. Hence, agent i's budget constraint becomes $x_i + z_i + \mu_i z_j = w_i$.

If both matching rates are positive, matching is bilateral or reciprocal, if only one matching rate is positive, matching is unilateral. If $\mu_1 = \mu_2 = 0$, we are in the standard case of voluntary public good provision without matching. If agent j is matched by the matching rate $\mu_i > 0$, her marginal rate of transformation between the private and the public good becomes $1+\mu_i$, i.e. her effective public good price falls to $\pi_j = \frac{1}{1+\mu_i} < 1$. (For a description of general matching mechanisms in a public good economy with an arbitrary number of agents see, e.g., [27].)

We now describe the feasible allocations and the effects of matching in the Kolm triangle ABC, which is an equilateral triangle with height W and identical side lengths $\overline{AB} = \overline{AC}$ $\overline{BC} = \frac{2}{\sqrt{3}}W$. Each point E in this triangle represents a feasible allocation (x_1, x_2, G) where x_1 is the length of the perpendicular from E to the side AB, x_2 is the length of the perpendicular from E to side AC and G is the length of the perpendicular from E to the triangle's base side BC. A general geometric fact then implies that the sum of these barycentric coordinates sum up to the height of the equilateral triangle, i.e. that $x_1 + x_2 + G = W$ holds and thus the feasibility constraint is satisfied and no resources are wasted.

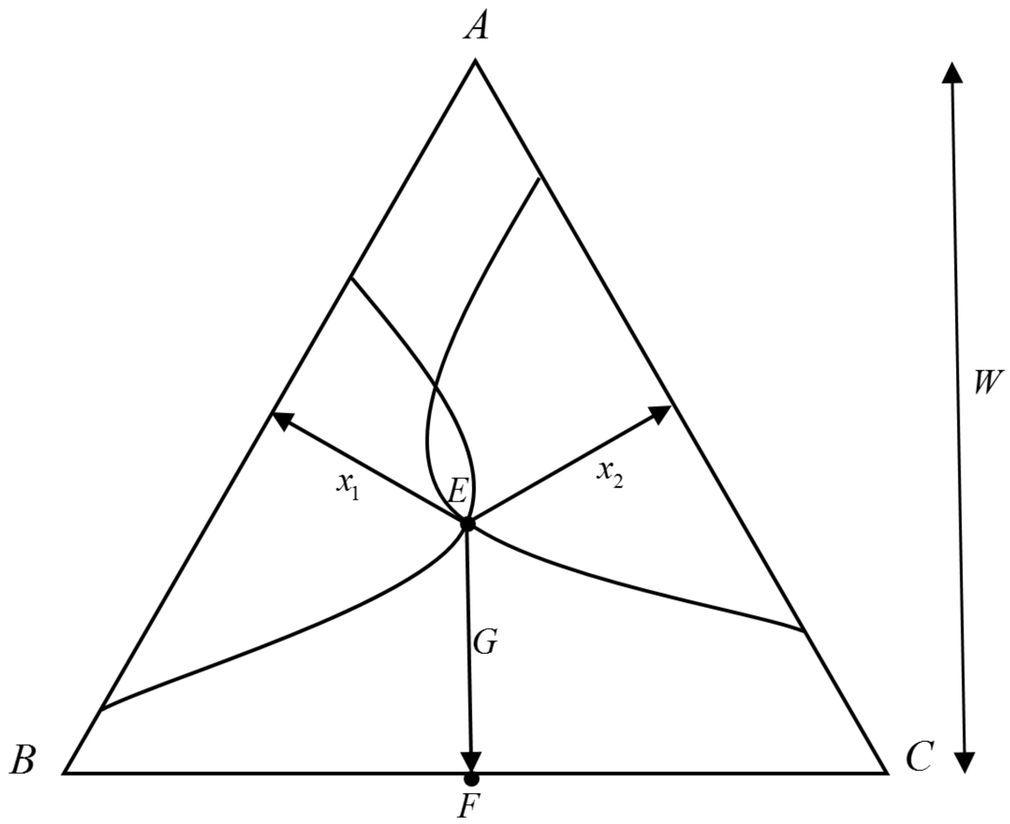

Figure 1: The Kolm triangle method

Each point F on the base line BC of the Kolm triangle represents a distribution of the total income between the two agents, i.e. $\frac{w_1}{w_2} = \frac{\overline{BF}}{\overline{FC}}$. Precisely, we have $\overline{BF} = \frac{2}{\sqrt{3}} w_1$ and $\overline{FC} = \frac{2}{\sqrt{3}} w_2$. Thus, a move of point F to the right indicates a redistribution of income from agent 2 to agent 1.

In the Kolm triangle, agent 1's indifference curves are bending downwards and convex, while the indifference curves of agent 2 are bending upwards and convex as depicted in Figure 1 for the indifference curves running through point E. (How the indifference curves in the Kolm triangle are obtained from the standard indifference curves in an x_1-G-diagram is described in detail by [4].)

3. Interior matching equilibria

After having described the Kolm triangle method in general we will use another Figure 2 to determine which flat contribution agent 1 will choose if her income is w_1 and she is matched by the other agent with the matching rate $\mu_2 \geq 0$. (For a consideration of distorted individual public good prices in the Kolm triangle see also [13], pp. 300 – 301.) The budget line, which then results for agent 1, is the straight line with slope $m(\mu_2) := -\sqrt{3}\frac{1+\mu_2}{1-\mu_2}$ running through the endowment point F (see the Appendix for an exact derivation of the budget lines in the Kolm triangle under matching). The point of tangency D between this budget line and an indifference curve of agent 1 then describes this agent's optimal allocation. There, agent 1's marginal rate of transformation $1+\mu_2$ as induced by the matching rate μ_2 equals this agent's marginal rate of substitution between the private and the public good.

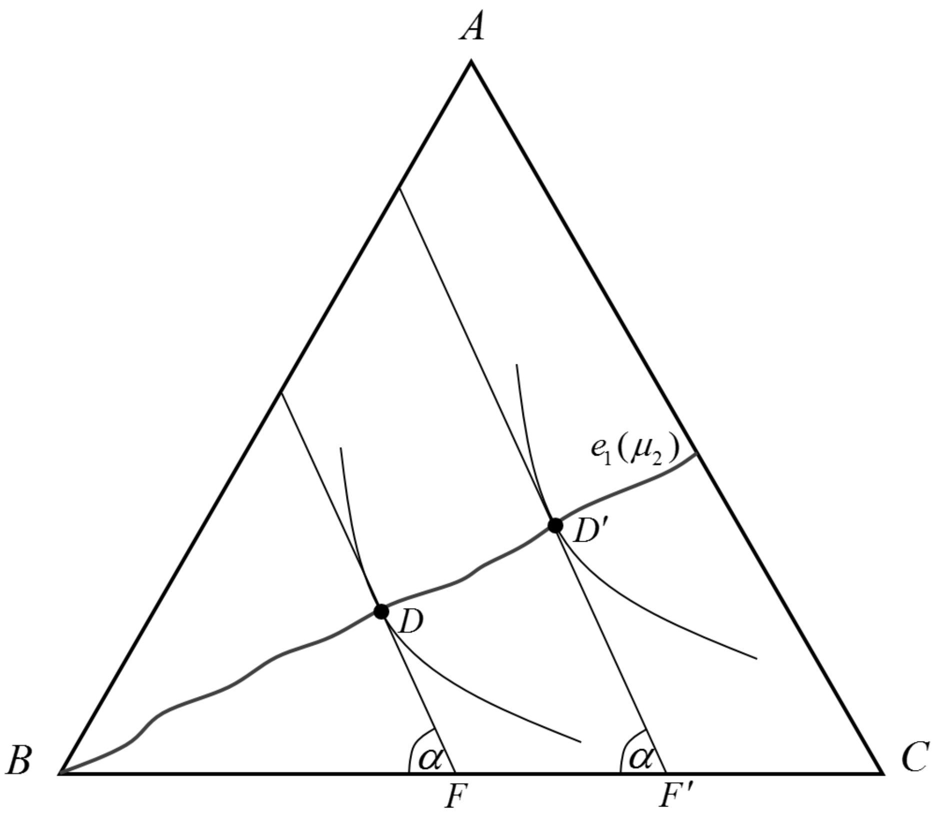

Figure 2: Expansion paths in the Kolm triangle

Now assume that agent 1's income w_1 is increased so that the budget line is shifted parallel to the right leading to the endowment point F'. As an implication of non-inferiority of the private and the public good, the optimum D' on the new budget line lies further away both from triangle side AB (which indicates higher demand of private consumption) and from the triangle side BC (which indicates higher demand of the public good by agent 1). If we now vary agent 1's income continuously and connect all optimal points for the given matching rate μ_2 we obtain agent 1's income expansion path $e_1(\mu_2)$. Along this path agent 1's indifference curves in the Kolm triangle all have the slope $m(\mu_2) = -\sqrt{3}\frac{1-\mu_2}{1+\mu_2}$, i.e. the slope of the budget lines given μ_2. Agent 1's expansion paths start in point B and, as follows from the explanation before, they are running northeast. The same consideration leads to expansion paths $e_2(\mu_1)$ for agent 2, which are running northwest in Figure 3.

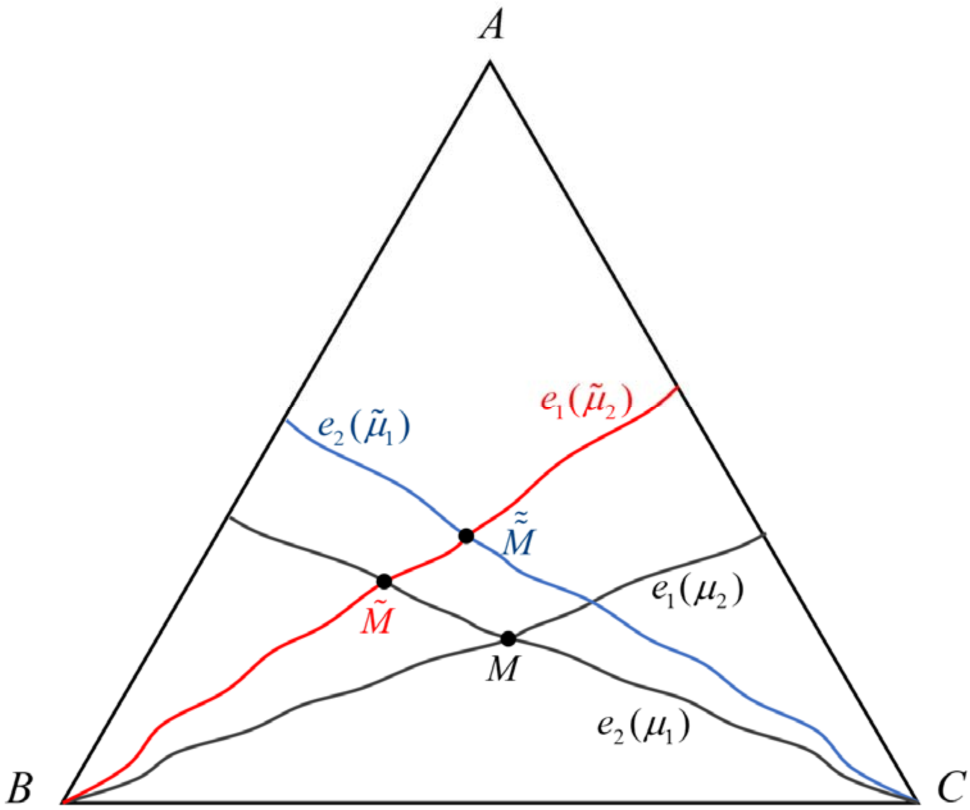

Figure 3: Matching equilibria

In a matching equilibrium both agents non-cooperatively choose their flat contributions in the face of the given matching rates μ_1 and μ_2, respectively. Hence, if agent i's flat contribution in a matching equilibrium is positive, she must be in a position where her marginal rate of substitution coincides with her marginal rate of transformation $1+\mu_2$ as implied by matching, which means that her equilibrium position must be on her expansion path $e_1(\mu_2)$. Therefore, if the matching equilibrium $M = M(\mu_1, \mu_2)$ for the matching rates μ_1 and μ_2 is interior with strictly positive flat contributions of both agents, it is represented in the Kolm triangle as the point of intersection between the expansion paths $e_1(\mu_2)$ and $e_2(\mu_1)$ (see Figure 3). By this argument, the basic idea of the Aggregative Game Approach as devised by [28] is integrated in the Kolm triangle.

It is now quite straightforward to describe in the Kolm triangle how an interior matching equilibrium is modified through a change of the matching rates. Assume that the rate at which agent 1's flat contributions are matched rises from the original $\mu_2 \geq 0$ to some $\tilde{\mu}_2 > \mu_2$. Then convexity of agent 1's indifference curves implies that her new expansion path $e_1(\tilde{\mu}_2)$ lies everywhere above $e_1(\mu_2)$. The point of intersection $\widetilde{M} = M(\mu_1, \tilde{\mu}_2)$ between $e_1(\tilde{\mu}_2)$ and $e_2(\mu_1)$, which characterizes the new matching equilibrium, thus is located on $e_2(\mu_1)$ north-west of the original matching equilibrium, so that $\widetilde{M}$ is further away from the triangle sides BC and AC. Therefore, it is obvious from Figure 3 that an increase of the matching rate μ_2 leads to an increase both of public good supply and private consumption of agent 2, which entails that agent 2 becomes better off. Private consumption of the matched agent 1 is reduced instead. A simultaneous increase of agent 1's matching rate μ_1 would clearly increase public good supply in the matching equilibrium still further, which in Figure 3 leads to the matching equilibrium $\widetilde{\widetilde{M}}$.

Starting from the Nash equilibrium without matching these considerations in particular show that in the case of interior matching equilibria the introduction of a matching mechanism mitigates the underprovision problem, actually the more the higher the matching rates are. However, with higher matching rates the range of income distributions, for which interior matching equilibria in fact result, is shrinking. Using the Kolm triangle, we will show this in the next section for the special case where utility functions and matching rates are identical for both agents. This specification allows us to focus on the impact the income distribution has on the matching equilibria, which is the central topic of this paper.

3. Interiority and the income distribution

In this section it is assumed that both agents have the same utility function $u(x_i, G)$ and that there is reciprocal matching with the uniform matching rate $\mu = \mu_1 = \mu_2$. The point of intersection $M(\mu)$ between the expansion paths $e_1(\mu)$ and $e_2(\mu)$, which represents the interior matching equilibrium in this case, then is located on the vertical height of the Kolm triangle (see Figure 4). The symmetric interior matching equilibrium, which results in this case, is characterized by identical private consumption levels $x(\mu)$ of both agents and public good supply $G(\mu)$. It follows from the analysis in Section 2 that $G(\mu)$ is increasing in the matching rate μ, so that the point $M(\mu)$ is moving upwards in the Kolm triangle when μ increases. Private consumption $x(\mu) = \dfrac{W - G(\mu)}{2}$ then clearly is falling in the matching rate μ.

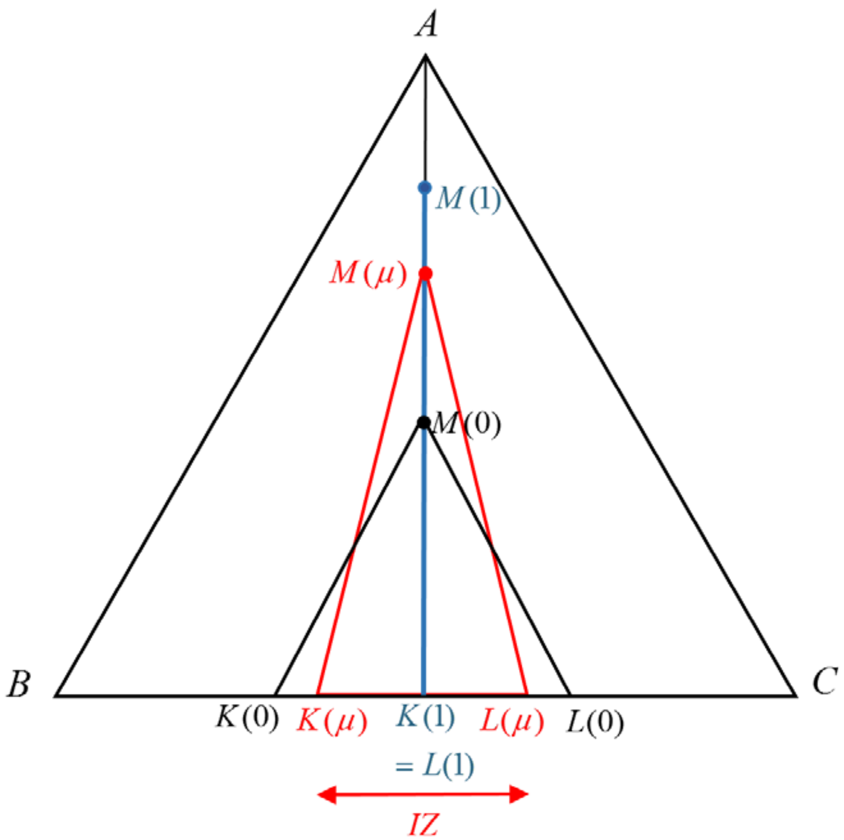

Figure 4: Interiority zones

We now show how the range of income distributions for which $M(\mu)$ actually is the interior matching equilibrium depends on μ. To this end, we first of all note that for some given matching rate μ, the minimum income $\underline{w}_a(\mu)$, at which an agent starts to make a positive flat contribution to the public good, is determined by

(1) $$\underline{w}_a(\mu) = x(\mu) + \frac{\mu G(\mu)}{1+\mu}.$$

This follows since an agent with income $\underline{w}_a(\mu)$ has the private consumption level $x(\mu)$ of the interior matching equilibrium and makes no positive flat contribution by her own. Rather, she spends the whole residual $\underline{w}_a(\mu) - x(\mu)$ for matching the other agent's flat contribution, which is $\frac{G(\mu)}{1+\mu}$. Then the matching agent is at the border of an interior solution, where nothing is left over for an own positive flat contribution. Inserting $x(\mu) = \frac{W - G(\mu)}{2}$ into (1) gives

(2) $$\underline{w}_a(\mu) = \frac{1}{2}(W - \frac{1-\mu}{1+\mu}G(\mu)).$$

Let $\overline{w}_a(\mu) = W - \underline{w}_a(\mu) = \frac{1}{2}(W + \frac{1-\mu}{1+\mu}G(\mu))$. Then the matching equilibrium $M(\mu)$ is attained for all income distributions (w_1, w_2) leading into the interval $\left[\underline{w}_a(\mu), \overline{w}_a(\mu)\right]$. Consequently, all income redistributions within this interval do not change the matching equilibrium.

In Figure 4, the "interiority zone" IZ for the given matching rate $\mu \in (0,1)$ is described by the interval $\left[K(\mu), L(\mu)\right]$ on the horizontal side of the Kolm triangle. The two straight lines $M(\mu)K(\mu)$ and $M(\mu)L(\mu)$ that delimit the interiority zone are the budget lines for agent 1 and agent 2, respectively, which result if either agent 1 or agent 2 has the income $\underline{w}_a(\mu)$ and the common matching rate is μ.

For $\mu = 0$, i.e. at the original Nash equilibrium without matching, we have $\underline{w}_a(0) = \frac{W - G(0)}{2} = x(0)$. In this case, the triangle $M(0)K(0)L(0)$ is equilateral. At $\mu = 1$, however,

$\underline{w}_a(1) = \overline{w}_a(1) = \frac{W}{2} > \underline{w}_a(0)$ results, so that $K(1)$ and $L(1)$ coincide and the interiority zone degenerates to a single point. The matching rate $\mu = 1$, however, is needed for attaining a Pareto optimal outcome in the case of a uniform matching rate since only then the both agents' marginal rates of substitution add up to the marginal rate of transformation, i.e. $\frac{2}{1+\mu} = 1$ holds so that the Samuelson condition for efficient pubic good supply is satisfied. This shows that attaining a Pareto optimal solution through matching only is possible for a very specific distribution of income.

The transition from $\mu = 0$ to $\mu = 1$ thus makes the neutrality zone smaller in a quite extreme sense, which reflects in a descriptive and intuitive manner a general result by [27]. For some further analysis we take the derivative of $\underline{w}_a(\mu)$ w.r.t. μ, for which we get by letting $G'(\mu) = \frac{\partial G}{\partial \mu}$ and $\underline{w}_a'(\mu) = \frac{\partial \underline{w}_a}{\partial \mu}$

(3) $\qquad \underline{w}_a'(\mu) > 0 \qquad$ if $\qquad (1-\mu)^2 G'(\mu) < 2G(\mu)$.

Since $G'(\mu) > 1$ is bounded from above on the closed interval $[0,1]$, condition (3) is clearly fulfilled if $\mu < 1$ is close to one so that monotonicity of $\underline{w}_a(\mu)$ is ensured in this region.

Condition (3) is fulfilled even on the entire interval $\mu \in [0,1]$ if, for instance, both agents have the same symmetric preferences of the Cobb-Douglas type, i.e. $u(x_i, G) = x_i G$. In this special case, the expansion paths are straight lines in the Kolm triangle and $G(\mu) = \frac{1+\mu}{3+\mu} W$ holds. Eq. (2) then gives $\underline{w}_a(\mu) = \frac{1+\mu}{3+\mu} W$, which is increasing for $\mu \in [0,1]$, and $\overline{w}_a(\mu) = \frac{2}{3+\mu} W$, which is decreasing in μ. In the two extreme cases with $\mu = 0$ and $\mu = 1$, we especially have $\underline{w}_a(0) = G(0) = x(0) = \frac{W}{3}$ and $\overline{w}_a(0) = \frac{2}{3} W$ for $\mu = 0$, and $\underline{w}_a(1) = \overline{w}_a(1) = G(1) = \frac{W}{2}$ and $x(1) = \frac{W}{4}$ for $\mu = 1$. For $\mu = \frac{1}{2}$ as an example of an intermediate value of the matching rate we get $\underline{w}_a(\frac{1}{2}) = \frac{3}{7} W = 0.43W$ and $\overline{w}_a(\frac{1}{2}) = \frac{4}{7} W = 0.57W$.

As long as the income distribution is in the interiority zone the participation constraint for the matching mechanism is automatically satisfied, i.e. for any matching rate μ with $0 < \mu \leq 1$ both agents are better off in the matching equilibrium $M(\mu)$ than in the original Nash equilibrium $M(0)$. This follows since in the symmetrical situation considered here both agents have higher utility in the Pareto optimal outcome $M(1)$ than in $M(0)$. Therefore, in Figure 4 $M(1)$ and henceforth the entire segment $M(0)M(1)$ of matching equilibria for $\mu \in [0,1]$ lies in the lens, which is made up by the agents' indifference curves through $M(0)$. This confirms that the participation constraint is satisfied for interior matching equilibria in the special case considered here. For corner solutions, in which only one agent makes a strictly positive flat contribution to the public good, this, however, no longer needs to be true even when preferences and matching rates are identical. This will be shown in the next section.

5. Corner solutions

We now assume that the income distribution lies outside the interiority zone, i.e. that agent 1's income w_1 exceeds $\overline{w}_a(\mu)$ and, consequently, agent 2's income w_2 falls below $\underline{w}_a(\mu)$. Then only agent 1 will make a positive flat contribution while agent 2 is only indirectly contributing to the public good by subsidizing agent 1. In Figure 5, the endowment point F then lies right to $L(\mu)$. The corner matching equilibrium $Q(\mu)$, which is obtained as the point of intersection between agent 1's budget line with slope $m(\mu) = -\sqrt{3}\frac{1-\mu}{1+\mu}$ with her expansion path $e_1(\mu)$, is located northeast to $M(\mu)$. In $Q(\mu)$ public good supply and utility of agent 1 hence are higher than in $M(\mu)$. (See [29], for an analysis of corner matching equilibria in the general case with n agents.)

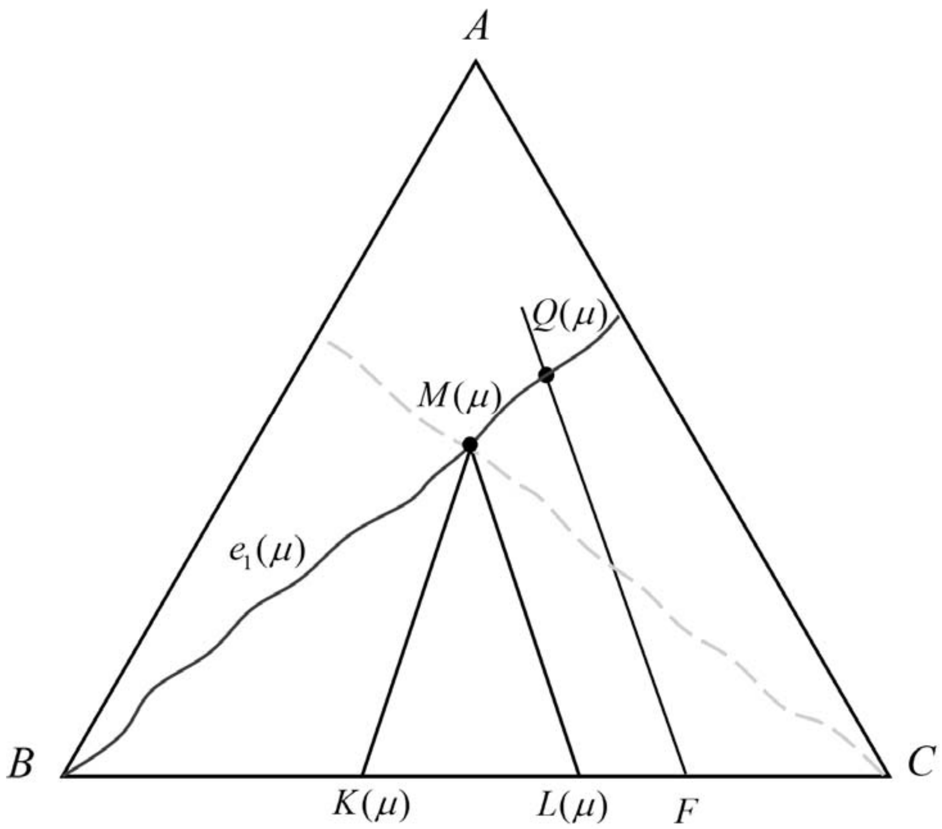

Figure 5: A corner equilibrium

Since agent 1 thus has higher utility in $M(\mu)$ than in the original Nash equilibrium $M(0)$ the participation constraint is automatically fulfilled for agent 1 in $Q(\mu)$. But if the deviation from the interiority zone is not too large, i.e. F is not too far away from $L(\mu)$, agent 2 is also better off in $Q(\mu)$ than in $M(0)$. By a continuity argument this simply follows since agent 2 is better off in $M(\mu)$ than in $M(0)$.

For an exact description of agent 2's participation incentives in the Kolm triangle let P_2 be the point, at which the expansion path $e_1(\mu)$ of agent 1 intersects agent 2's indifference curve through $M(0)$ (see Figure 6). Agent 2 then is better off in the corner matching equilibrium than in the standard Nash equilibrium when the matching equilibrium lies on the segment $M(\mu)P_2$ of $e_1(\mu)$, which results when the endowment point is located between $L(\mu)$ and $\hat{L}(\mu)$. If, however, the income of agent 1 is so high (and agent 2's income correspondingly so low) that the matching equilibrium lies north-east of P_1, agent 2 attains lower utility in the matching equilibrium than in the original Nash equilibrium. Then the participation constraint is violated. The participation incentives for agent 1 are described by a completely analogous reasoning, which leads to the point P_1 on agent 2's expansion path $e_2(\mu)$ and to the point $\hat{K}(\mu)$ on the triangle side BC. Consequently, the range of income distributions for which the participation constraint holds for both agents thus is visualized in Figure 6 by the interval $\hat{K}(\mu)\hat{L}(\mu)$ on the triangle side BC, which corresponds to an interval of income distributions $\left[\underline{w}_b(\mu), \overline{w}_b(\mu)\right]$.

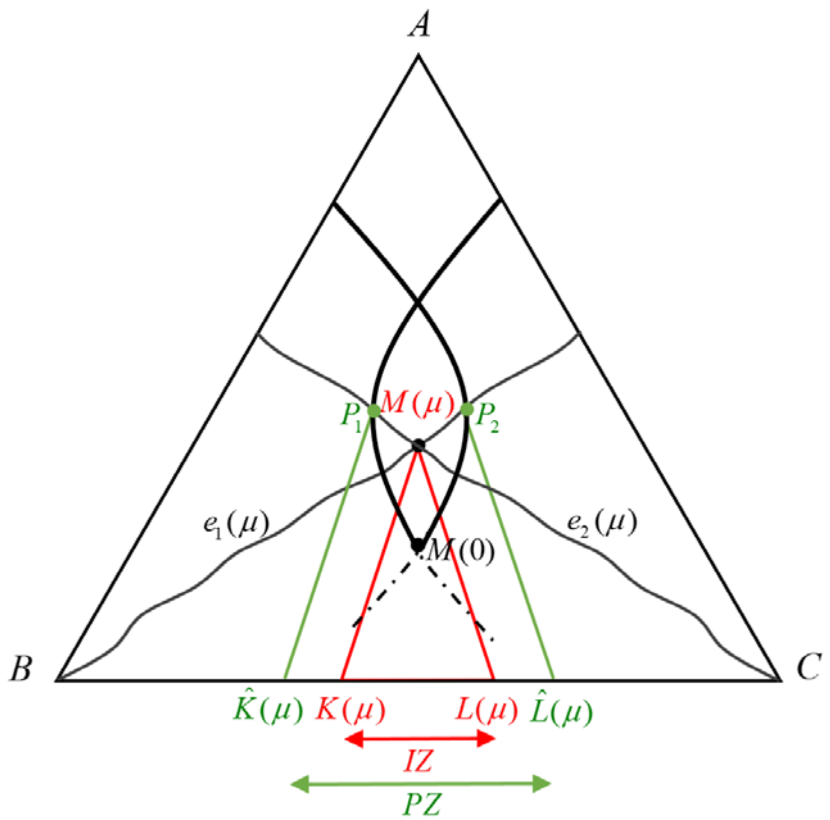

Figure 6: Comparison between the interiority and the participation zone

It is a direct consequence of the monotonicity of expansion paths that P_1 lies to the left and P_2 to the right of $M(\mu)$. This implies that for any given $\mu \in (0,1]$ the "participation zone" $PZ = \hat{K}(\mu)\hat{L}(\mu)$ is larger than the interiority zone $IZ = K(\mu)L(\mu)$, i.e. $\underline{w}_b(\mu) < \underline{w}_a(\mu)$ and $\overline{w}_b(\mu) > \overline{w}_a(\mu)$ holds. Hence, the requirements for income distributions for satisfying interiority of a matching equilibrium are more challenging than the requirements for satisfying the participation constraint.

In the special case where both agents have the same Cobb-Douglas utility function $u(x_i, G) = x_i G$ for a given μ we can explicitly determine the critical value $\overline{w}_b(\mu)$ for any $\mu \in (0,1]$ from the condition

(4) $$(W - \overline{w}_b(\mu) - \mu \frac{\overline{w}_b(\mu)}{2})(1+\mu)\overline{w}_b(\mu) = \frac{W^2}{9}.$$

The left hand side of (4) gives agent 2's utility when she is in P_2 (where her income is $w_2 = W - \overline{w}_b(\mu)$), where she matches agent 1's flat contribution $\frac{\overline{w}_b(\mu)}{2}$ with the matching rate μ while the right hand side gives agent 2's utility in the Nash equilibrium without matching where $x(0) = G(0) = \frac{W}{3}$. Solving the quadratic equation in (4) yields

(5) $$\overline{w}_b(\mu) = \frac{W}{2+\mu}(1 + \frac{1}{3}\sqrt{\frac{1+5\mu}{1+\mu}}) \quad \text{and} \quad \underline{w}_b(\mu) = \frac{W}{2+\mu}(1 + \mu - \frac{1}{3}\sqrt{\frac{1+5\mu}{1+\mu}}).$$

For the special value of intermediate matching $\mu = \frac{1}{2}$ we in particular have $\underline{w}_b(\frac{1}{2}) \approx 0.4W$ and $\overline{w}_b(\frac{1}{2}) \approx 0.6W$. This confirms that in this example the participation zone is larger than the interiority zone as graphically described by Figure 6 for the general case.

It generally holds that for the matching rate $\mu = 1$ the participation zone $\hat{K}(1)\hat{L}(1)$ is a non-degenerate interval while the interiority zone collapses into a straight line in this case.

6. Conclusion

It is well-known that in a public goods economy matching of public good contributions can successfully be employed to increase public good supply and to achieve a Pareto improvement over the conventional non-cooperative Nash equilibrium. In this paper, it has been shown how the Kolm triangle method can be used to visualize important effects of matching in an elegant way, so basically the increase of public good supply through matching – be it unilateral or reciprocal. We describe in a quite intuitive way, how in a special situation (with identical preferences and identical matching rates) the interiority of matching equilibria depends on the income distribution and especially, how the "interiority zone", i.e. the range of income distributions leading to an interior matching equilibrium, is shrinking when the matching rate increases. Moreover, we were able to delimit the "participation zone" in the Kolm triangle, i.e. the set of income distributions, for which the matching equilibrium becomes Pareto-superior to the original Nash equilibrium so that agents voluntarily enter the matching scheme. In this context, an important and novel insight has been that the participation zone is larger than the interiority zone, which means that also corner matching equilibria in which only one agent makes a positive flat contribution to the public good may make both agents better off. How this welfare effect can be generalized to the case of different utility functions and matching rates will be an issue of future research.

Appendix

We assume that agent 1 contributes some $d \in (0, w_1]$ units of her endowment w_1 to the public good. Then, her new position must be on a parallel to the Kolm triangle side AB whose distance to the endowment point F is d. Given the matching rate $\mu_2 \geq 0$, agent 1's new position must lie on a parallel to BC at the same time, which is $(1+\mu_2)d$ units away from this side of the Kolm triangle. The new position thus is given as the point of intersection between these two parallels, which is point T in Figure 7.

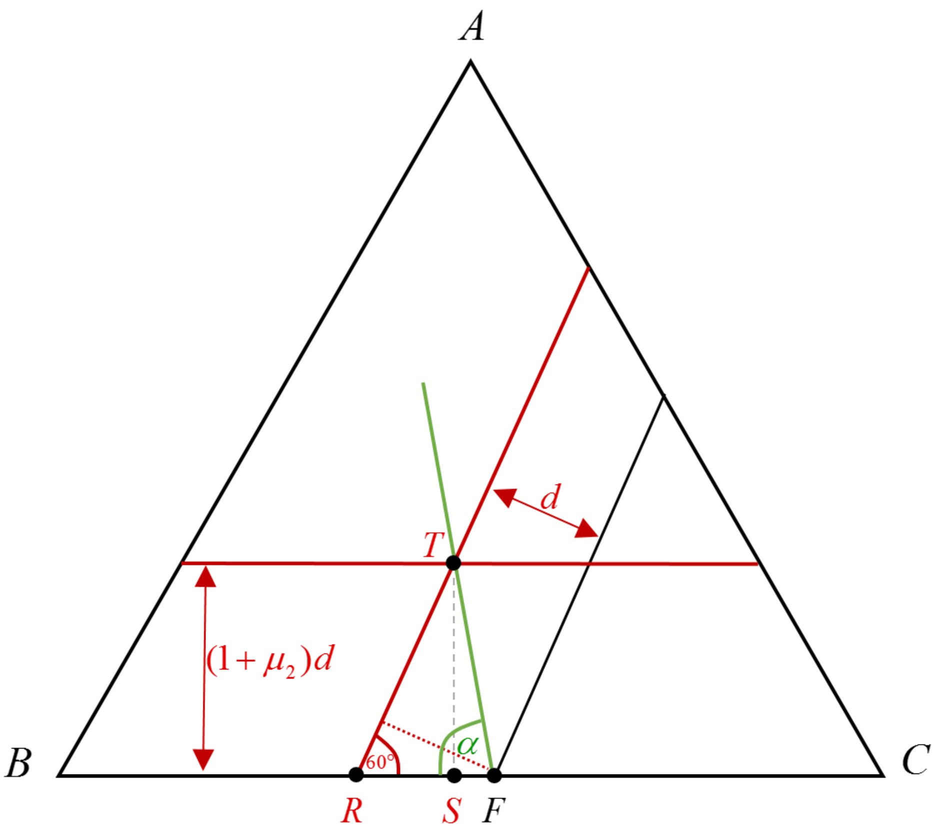

Figure 7: The slope of budget lines under matching

The budget line for the given matching rate μ_2 then is the straight line FT in Figure 7 (on which point D of Figure 2 in the main text indicates agent 1's optimal choice). To determine the slope $m(\mu_2)$ of this budget line we first note that $\overline{RF} = \frac{d}{\sin 60°} = \frac{2d}{\sqrt{3}}$ and $\overline{RS} = \frac{(1+\mu_2)d}{\tan 60°} = \frac{(1+\mu_2)d}{\sqrt{3}}$, which gives $\overline{SF} = \overline{RF} - \overline{RS} = \frac{(1-\mu_2)d}{\sqrt{3}}$. Then $m(\mu_2) = -\tan\alpha = -\frac{\overline{ST}}{\overline{SF}} = -\sqrt{3}\frac{1+\mu_2}{1-\mu_2}$.

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