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Efficient Random Assignment under a Combination of Ordinal and Cardinal Information on Preferences

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Summary

Consider a collection of *m* indivisible objects to be allocated to n agents, where $m \geq n$. Each agent falls in one of two distinct categories: either he (a) has a complete ordinal ranking over the set of individual objects, or (b) has a set of "plausible" benchmark von Neumann-Morgenstern (vNM) utility functions in whose non-negative span his "true" utility is known to lie. An allocation is undominated if there does not exist a preference-compatible profile of vNM utilities at which it is Pareto dominated by another feasible allocation. Given an undominated allocation, we use the tools of linear duality theory to construct a profile of vNM utilities at which it is ex-ante welfare maximizing. A finite set of preference-compatible vNM utility profiles is exhibited such that every undominated allocation is ex-ante welfare maximizing with respect to at least one of them. Given an arbitrary allocation, we provide an interpretation of the constructed vNM utilities as subgradients of a function which measures worst-case domination.

Keywords: Random Assignment, Efficiency, Duality, Linear Programming

JEL Classification: C61, D01, D60

I am indebted to two anonymous referees and an associate editor for thoughtful comments that improved the quality and enhanced the scope of this work. I am grateful to Vikram Manjunath for many insightful conversations and for his detailed comments on an earlier version of the manuscript. I thank Mihai Manea and Jay Sethuraman for their helpful comments on an earlier version of this work.

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Efficient random assignment under a combination of ordinal and cardinal information on preferences

Stergios Athanassoglou*

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Abstract

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1 Introduction

In an influential paper, Bogomolnaia and Moulin [4] consider the probabilistic assignment of n indivisible objects to n agents. Agents are endowed with strict ordinal preferences over the set of objects. Objects are assigned via lotteries, which, in light of the Birkhoff-von Neumann theorem [3], can be represented by arrays of probabilities. To accommodate this probabilistic environment, Bogomolnaia and Moulin [4] adapt the familiar notion of Pareto efficiency to random assignments by introducing the concept of *ordinal efficiency.* A random assigment is ordinally efficient if agents cannot trade probability shares of objects to achieve a new random allocation that stochastically dominates the original one. Bogomolnaia and Moulin show that ordinal efficiency is equivalent to the acyclicity of a particular kind of binary relation between objects.[1] In a later contribution, Abdulkadiroglu and Sonmez [1] provide a different characterization of ordinal efficiency based on a novel concept of dominated sets of assignments. In recent years, ordinal efficiency has been seen as an important benchmark in random assignment and has motivated the study and comparison of individual allocation mechanisms (Manea [9], Manea [11], Kesten [8], Che and Kojima [6]).

McLennan [12] offers a different characterization of ordinal efficiency. He considers the weak preference domain and shows that an allocation is ordinally efficient if and only if it is ex-ante welfare maximizing at some profile of von Neumann-Morgenstern (vNM) utilities, which is compatible with the underlying ordinal preferences. In his proof, he establishes and uses a new version of the separating hyperplane theorem. Manea [10] provides a simpler, constructive proof of McLennan's result that is based on the acyclicity of the binary relation discussed in [4] and [7]. The constructed profile of vNM utilities is related to a given weak representation of this (acyclic) binary relation.

In an important recent contribution Carroll [5] extends McLennan's characterization to economic environments in which agents' preferences are incompletely known, so that an agent i's vNM utility function is only assumed to lie in nonempty, convex, and relatively open sets U_i. He shows that if an allocation is undominated (meaning that there exists no allocation that ex-ante dominates it for all plausible utility functions), then this allocation is ex-ante welfare maximizing at some vNM utility functions $u_i \in U_i$. Similar to McLennan, Carroll employs a hyperplane-separation line of reasoning and focuses on proving the existence of these utility functions (without actually exhibiting them).

Contribution. Our own economic environment is more general than McLennan's and considerably less than Carroll's. Each agent i falls in one of two distinct categories: either he (a) has a complete ordinal ranking over the set of individual objects, or (b) has a set of "plausible" benchmark vNM utility functions (representing, say, different states of nature) in whose non-negative span

[1] Katta and Sethuraman [7] extend Bogomolnaia and Moulin's analysis to the weak preference domain.

his “true” utility is known to lie.[2] In this context, a vNM utility profile is said to be *preference-compatible* if its individual vNM utility functions are consistent with all available ordinal information for the first category of agents, and lie in the non-negative span of the benchmark utility functions of the latter.

Suppose an allocation is *undominated*, meaning that that there is no other allocation that is guaranteed to Pareto dominate it for *all* preference-compatible utility profiles. Then, using the tools of linear duality theory, a preference-compatible utility profile is constructed at which this allocation is ex-ante welfare maximizing. When there is full ordinal and no cardinal information, this result recovers the ordinal efficiency theorem due to McLennan [12]. Consequently, a finite set of preference-compatible vNM profiles is exhibited such that every undominated allocation is ex-ante welfare maximizing with respect to at least one of them. A combinatorial upper bound is given on the cardinality of this set. Given an arbitrary allocation, we provide an interpretation of the constructed vNM utilities as subgradients of a function which measures worst-case domination.

It is our hope that the simplicity of our LP-based approach may prove helpful in thinking about related problems in the growing field of random assignment.

Structure of the Paper. The structure of the paper is as follows. Section 2 introduces the model, and Section 3.1 provides proofs of our main results based on LP duality. Section 3.2 generalizes the approach pursued in Section 3.1 to arbitrary (i.e., not necessarily undominated) allocations and offers an interpretation of the constructed vNM utility profiles as subgradients of a function measuring worst-case domination. Section 4 provides concluding remarks.

2 Model Description

Consider an economy with a set N of n agents and M of m objects indexed by $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$, respectively. Suppose without loss of generality that $m \geq n$, allowing for the possibility of “dummy” objects that correspond to not being assigned anything at all.

Agents are partitioned in two groups N^O and N^C. Agents belonging in N^O have ordinal preferences over the set of objects that are expressed by the complete, reflexive and transitive relation $\succeq_i$. Hence, if objects j_1 and j_2 are such that $j_1 \succeq_i j_2$ and $j_2 \succeq_i j_1$ then agent i is indifferent between them, and this is denoted by $j_1 \sim_i j_2$. If $j_1 \succeq_i j_2$, but $j_2 \not\succeq_i j_1$, then agent i strictly prefers object j_1 to j_2, and this is denoted by $j_1 \succ_i j_2$.

Agents in N^C have no such clear-cut ordinal information over preferences. Instead, they have a

[2] I am thankful to an anonymous referee for suggesting the adoption of a more general preference environment than one that (solely) features complete ordinal preferences.

set of plausible benchmark vNM utility functions $\{u_i^l : M \mapsto \Re, \;\; l \in \{1, 2, ..., n_i\}\}$ in whose *non-negative span* their "true" utility function is known to lie. In other words, they know that their true utility will be equal to a non-negative linear combination of the benchmark ones. These sets of vNM utilities, which fit the far more general framework of Carroll [5], provide a way of imposing structure on the utility functions that need to be considered.

In what follows, a prime symbol following a given (column) vector denotes the vector's transpose. Throughout, we suppress the explicit dependence of our analysis on the economy's preferences. An *individual allocation* for agent i is a non-negative column vector $p_i = (p_{i1}, p_{i2}, \ldots, p_{im})'$ such that $\sum_j p_{ij} = 1$. An *allocation* $\mathbf{p} = (p_1', p_2', ..., p_n')'$ is a concatenation of a set of individual allocations p_i for $i = 1, 2, ..., n$ that satisfies $\sum_i p_{ij} \leq 1$ for all $j \in M$.[3] Let P denote the set of all allocations.

Depending on whether an agent i belongs to N^O or N^C, an individual allocation p_i *dominates* another q_i, whenever

$$i \in N^O : \sum_{a \succeq_i j} p_{ia} \geq \sum_{a \succeq_i j} q_{ia}, \text{ for all } j \in M, \text{ or}$$
$$i \in N^C : \sum_{a \in M} u_i^l(a) p_{ia} \geq \sum_{a \in M} u_i^l(a) q_{ia}, \text{ for all } l \in \{1, 2, .., n_i\}.$$

If at least one of the above inequalities is strict, then p_i *strictly dominates* q_i. The dominance relation defined on an individual allocation extends to its economy-wide equivalent in a natural way: an allocation $\mathbf{p}$ *dominates* an allocation $\mathbf{q}$ if p_i dominates q_i for every agent i; $\mathbf{p}$ *strictly dominates* $\mathbf{q}$ if $\mathbf{p}$ dominates $\mathbf{q}$, and if p_i strictly dominates q_i for some agent i. An allocation $\mathbf{p}$ is said to be *undominated* if there does not exist an allocation $\mathbf{q}$ that strictly dominates it.

Adhering to our previous discussion, a profile of von Neumann-Morgenstern (vNM) utility functions $\tilde{\mathbf{u}} = (\tilde{u}_i : M \rightarrow \Re, \;\; i \in N)$ is said to be *preference-compatible* if

$$i \in N^O : \;\; \tilde{u}_i(j_1) \geq \tilde{u}_i(j_2) \Leftrightarrow j_1 \succeq_i j_2, \;\; j_1, j_2 \in M, \;\; \text{and}$$
$$i \in N^C : \;\; \tilde{u}_i(j) = \sum_{l=1}^{n_i} \tilde{w}_i^l u_i^l(j), \;\; j \in M, \;\; \text{for some } \tilde{w}_i^l \geq 0$$

Finally, an allocation $\mathbf{p}$ is *ex-ante welfare maximizing* at a profile of vNM utilities $\tilde{\mathbf{u}}$ if it maximizes the social welfare function

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} \tilde{u}_i(j),$$

over the set of feasible allocations.

An example. To illustrate an instance of our model, consider an economy with three agents (1, 2 and 3) and three objects (a, b and c), in which $N^O = \{1, 2\}$ and $N^C = \{3\}$. Agent 1 strictly

[3] Note how the elements p_{ij} of $\mathbf{p}$ are positioned in lexicographic order. For reasons that will become apparent in Section 3, we avoid the more common representation of an allocation as a sub-stochastic matrix.

prefers object a to b and b to c. Agent 2 strictly prefers b to c and c to a. Agent 3 has no ordinal information on his preferences; instead he is aware of three different benchmark vNM utilities that correspond to his preferences in three plausible different states of the world (u_3^1, u_3^2, u_3^3), where

$$u_3^1 = (100, 5, 1), \quad u_3^2 = (1, 50, 90), \quad u_3^3 = (43, 45, 44).$$

Clearly, agent 3 cannot determine a definitive ordinal ranking from the above information. Instead, he simply knows that his true utility, u_3, will be some non-negative combination of the benchmarks u_3^1, u_3^2, and u_3^3, so that it lies in the set

$$\{u_3 : \;\; w_3^1 u_3^1 + w_3^2 u_3^2 + w_3^3 u_3^3, \;\; w_3^1 \geq 0, \; w_3^2 \geq 0, \; w_3^3 \geq 0\},$$

3 An Efficiency Theorem

3.1 The Main results

In what follows, we use LP duality to prove the paper's main result.

Theorem 1 *Suppose* $\hat{\mathbf{p}}$ *is undominated. There exists a preference-compatible profile of vNM utilities at which* $\hat{\mathbf{p}}$ *is ex-ante welfare maximizing.*

Proof. Consider an allocation $\hat{\mathbf{p}} \in P$. Where applicable, let $\mathbf{0}$ denote a zero vector of appropriate dimension. Consider the following linear program (LP) in standard form:

$$\begin{aligned}
\min_{\mathbf{p},\mathbf{r},\mathbf{q},\mathbf{s}} \quad & -\Big(\sum_{i \in N^O} \sum_{j=1}^{m} r_{ij} + \sum_{i \in N^C} \sum_{l=1}^{n_i} q_{il} \Big) \\
\text{subject to:} \quad & \sum_{k \succeq_i j} p_{ik} - r_{ij} = \sum_{k \succeq_i j} \hat{p}_{ik}, \;\; j \in M, \;\; i \in N^O \\
& \sum_{j=1}^{m} u_i^l(j) p_{ij} - q_{il} = \sum_{j=1}^{m} u_i^l(j) \hat{p}_{ij}, \;\; l \in \{1, 2, ..., n_i\}, \;\; i \in N^C \\
& \sum_{j=1}^{m} p_{ij} + s_j = 1, \;\; i \in N \\
& \sum_{i=1}^{n} p_{ij} = 1, \;\; j \in M \\
& \mathbf{p} \geq \mathbf{0}, \;\; \mathbf{r} \geq \mathbf{0}, \;\; \mathbf{q} \geq \mathbf{0}, \;\; \mathbf{s} \geq \mathbf{0}.
\end{aligned} \tag{1}$$

By definition, the solution $(\mathbf{p}, \mathbf{r}, \mathbf{q}, \mathbf{s}) = (\hat{\mathbf{p}}, \mathbf{0}, \mathbf{0}, \hat{\mathbf{s}})$, where $\hat{s}_j = 1 - \sum_{i \in N} \hat{p}_{ij}$, is feasible and establishes an upper bound of 0 for the problem's optimal cost (i.e., objective value).

Using the definition of domination, it is easy to see that $\hat{\mathbf{p}}$ is undominated if and only if the optimal solution $(\mathbf{p}^*, \mathbf{r}^*, \mathbf{q}^*, \mathbf{s}^*)$ of the primal problem (1) is equal to $(\hat{\mathbf{p}}, \mathbf{0}, \mathbf{0}, \hat{\mathbf{s}})$, thus yielding an optimal cost of 0.

Taking the dual of (1), and letting $\mathbf{1}$ denote a unit vector of appropriate dimension, we obtain[4]

$$\begin{aligned}
\max_{\mathbf{x},\mathbf{w},\mathbf{y},\mathbf{z}} \quad & \sum_{i\in N^O}\sum_{j=1}^{m} x_{ij}\sum_{k\succeq_i j}\hat{p}_{ik} + \sum_{i\in N^C}\sum_{l=1}^{n_i}\sum_{j=1}^{m} w_{il}u_i^l(j)\hat{p}_{ij} + \sum_{i=1}^{n} y_i + \sum_{j=1}^{m} z_j \\
\text{subject to:} \quad & \sum_{k\preceq_i j} x_{ik} + y_i + z_j \le 0, \;\; j\in M, \;\; i\in N^O \\
& \sum_{l=1}^{n_i} u_i^l(j)w_{il} + y_i + z_j \le 0, \;\; j\in M, \;\; i\in N^C \\
& \mathbf{x}\ge \mathbf{1}, \;\; \mathbf{w}\ge \mathbf{1} \\
& \mathbf{y} \text{ free variable}, \;\; \mathbf{z}\ge \mathbf{0}.
\end{aligned} \tag{2}$$

By strong duality (see Theorem 4.4 in [2]), the primal problem has an optimal cost of 0 (which, as mentioned before, is equivalent to $\hat{\mathbf{p}}$ being undominated) if and only if the optimal solution of the dual problem (2), $(\hat{\mathbf{x}},\hat{\mathbf{w}},\hat{\mathbf{y}},\hat{\mathbf{z}})$,[5] satisfies

$$\sum_{i\in N^O}\sum_{j=1}^{m} \hat{x}_{ij}\sum_{k\succeq_i j}\hat{p}_{ik} + \sum_{i\in N^C}\sum_{l=1}^{n_i}\sum_{j=1}^{m} \hat{w}_{il}u_i^l(j)\hat{p}_{ij} + \sum_{i=1}^{n} \hat{y}_i + \sum_{j=1}^{m} \hat{z}_j = 0. \tag{3}$$

Now, let $\hat{\mathbf{u}}$ denote a profile of von-Neumann Morgenstern (vNM) utilities such that, for all $j\in M$,

$$\begin{aligned}
i\in N^O: \;\; \hat{u}_i(j) &= \sum_{k\preceq_i j}\hat{x}_{ik}, \\
i\in N^C: \;\; \hat{u}_i(j) &= \sum_{l=1}^{n_i} u_i^l(j)\hat{w}_{il},
\end{aligned} \tag{4}$$

Recall that since $(\hat{\mathbf{x}},\hat{\mathbf{w}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ is feasible, we must have $(\hat{\mathbf{x}},\hat{\mathbf{w}})\ge\mathbf{1}$. In combination with Eqs. (4), this immediately establishes that $\hat{\mathbf{u}}$ is preference-compatible. (Note how when an agent $i\in N^O$ is indifferent between two objects, they are assigned equal utility.) Rearranging terms, Eq. (3) can be rewritten in the following way

$$\begin{aligned}
& \sum_{i\in N^O}\sum_{j=1}^{m} \hat{x}_{ij}\sum_{k\succeq_i j}\hat{p}_{ik} + \sum_{i\in N^C}\sum_{l=1}^{n_i}\sum_{j=1}^{m} \hat{w}_{il}u_i^l(j)\hat{p}_{ij} = -\left(\sum_{i=1}^{n}\hat{y}_i + \sum_{j=1}^{m}\hat{z}_j\right) \\
\Rightarrow \;\; & \sum_{i=1}^{n}\sum_{j=1}^{m}\hat{u}_i(j)\hat{p}_{ij} = -\left(\sum_{i=1}^{n}\hat{y}_i + \sum_{j=1}^{m}\hat{z}_j\right)
\end{aligned} \tag{5}$$

Again by dual feasibility we must have

$$0\le \hat{u}_i(j)\le -(\hat{y}_i+\hat{z}_j) \tag{6}$$

Now, consider an arbitrary $\mathbf{p}\in P$. We have

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{m}\hat{u}_i(j)p_{ij} \;&\overset{(6)}{\le}\; \sum_{i=1}^{n}\sum_{k=1}^{m} -(\hat{y}_i+\hat{z}_j)p_{ij} \overset{(1)\,(2)}{\le} -\left(\sum_{i=1}^{n}\hat{y}_i + \sum_{j=1}^{m}\hat{z}_j\right) \\
&\overset{(5)}{=}\; \sum_{i=1}^{n}\sum_{j=1}^{m}\hat{u}_i(j)\hat{p}_{ij}.
\end{aligned} \tag{7}$$

[4] For details see Chapter 4.2 in Bertsimas and Tsitsiklis [2].

[5] The "hat" notation is adopted to denote the dependence of the optimal dual variables on $\hat{\mathbf{p}}$.

This observation concludes the proof. ■

Thus, we have constructed a preference-compatible utility profile $\hat{\mathbf{u}}$ at which the undominated allocation $\hat{\mathbf{p}}$ is ex-ante welfare maximizing. Before we prove our next result we focus on the feasible region of the dual LP (2) and note that no variable y_i can ever be strictly positive. Thus, we can safely impose the constraint $\mathbf{y} \leq \mathbf{0}$ so that the dual feasible region is substituted by the following polyhedron:

$$\begin{aligned}
&\sum_{k \preceq_i j} x_{ik} + y_i + z_j \leq 0, \quad j \in M, \quad i \in N^O \\
&\sum_{l=1}^{n_i} u_i^l(j) w_{il} + y_i + z_j \leq 0, \quad j \in M, \quad i \in N^C \\
&\mathbf{x} \geq \mathbf{1}, \quad \mathbf{w} \geq \mathbf{1} \\
&\mathbf{y} \leq \mathbf{0}, \quad \mathbf{z} \geq \mathbf{0}. \qquad (8)
\end{aligned}$$

We now provide a definition for the *extreme points* of a polyhedron.

Definition 1 *An* extreme point *of a polyhedron* Π *is a vector* $\mathbf{v_1} \in \Pi$ *such that we cannot find two vectors* $\mathbf{v_1}, \mathbf{v_2} \in \Pi$*, both different from* $\mathbf{v_1}$*, and a scalar* $\lambda \in [0,1]$ *that satisfy* $\mathbf{v_1} = \lambda \mathbf{v_2} + (1-\lambda)\mathbf{v_3}$.

In other words, an extreme point is an element of a polyhedron which cannot be written as a convex combination of two other elements (of the polyhedron), which are both distinct from it. Extreme points will play an important role in our next result.[6]

Theorem 2 *There exists a finite set of preference-compatible utility profiles such that every undominated allocation is ex-ante welfare maximizing for at least one of this set's elements. This set has cardinality no more than the number of extreme points of polyhedron (8).*

Proof. The set of (updated) dual constraints $\{\mathbf{x} \geq 1, \mathbf{w} \geq \mathbf{1}, \mathbf{y} \leq \mathbf{0}, \mathbf{z} \geq \mathbf{0}\}$, ensures that the dual feasible region (8) is a polyhedron that does not contain a line (see Definition 2.12 in [2]). Consequently, Theorem 2.6 of [2] implies that the dual feasible region contains at least one extreme point.

Let $\mathbf{E}$ denote the set of all extreme points of (8). By Theorem 2.3 and Corollary 2.1 in [2] its cardinality is bounded above by $\binom{a}{b}$, where a represents the number of constraints and b the number of variables of polyhedron (8). For every extreme point $\mathbf{e} = (\mathbf{x^e}, \mathbf{w^e}, \mathbf{y^e}, \mathbf{z^e}) \in \mathbf{E}$, introduce a vNM utility profile $\mathbf{u^e}$ consistent to Eq. (4) such that

$$i \in N^O: \quad u_i^{\mathbf{e}}(j) = \sum_{k \preceq_i j} x_{ik}^{\mathbf{e}},$$

[6] I am grateful to an anonymous referee for drawing attention to this result.

$$i \in N^C: \quad u_i^{\mathbf{e}}(j) = \sum_{l=1}^{n_i} u_i^l(j) w_{il}^{\mathbf{e}}. \tag{9}$$

Now, let $\hat{\mathbf{P}}$ denote the set of undominated allocations and consider an arbitrary $\hat{\mathbf{p}} \in \hat{\mathbf{P}}$ and the associated dual LP (2). By Theorem 2.8 of [2], the optimum of this dual LP must be attained at some extreme point $\mathbf{e}$ of (8). Consider the vNM utility profile $\mathbf{u}^{\mathbf{e}}$ as given by Eq. (9) for extreme point $\mathbf{e}$. Following the logic of Theorem 1, the allocation $\hat{\mathbf{p}}$ will be ex-ante welfare maximizing at $\mathbf{u}^{\mathbf{e}}$. Noting that the feasible region of LP (2) (represented by polyhedron (8)), and therefore its finite set of extreme points, is unaltered by changes in the dual's objective function, and repeating our argument for any element of $\hat{\mathbf{P}}$ establishes the result. ■

Remarks. The equivalence between extreme points and *basic feasible solutions* of a polyhedron (Theorem 2.3 in [2]) provides algebraic insight into the structure of the extreme points of (8) and gives a sense of their total number.[7] Indeed, applying this equivalence result to the special structure of polyhedron (8),in order to actually count its extreme points is an interesting combinatorial exercise in its own right. Having said this, the bound of Theorem 2 can be made considerably tighter when we consider that we can safely desregard extreme points $\mathbf{e}$ that satisfy

$$\sum_{k \preceq_i j} x_{ik}^{\mathbf{e}} + y_i^{\mathbf{e}} + z_j^{\mathbf{e}} < 0, \quad \text{for all } j \in M, \quad \text{for some } i \in N^O, \text{ and/or}$$
$$\sum_{l=1}^{n_i} u_i^l(j) w_{il}^{\mathbf{e}} + y_i^{\mathbf{e}} + z_j^{\mathbf{e}} < 0, \quad \text{for all } j \in M, \quad \text{for some } i \in N^C.$$

This is because feasible points satisfying the above conditions cannot, by first principles, be optimal: increasing the x_{ik} or w_{il} variables until one of the above constraints binds will increase the objective function value without resulting in infeasibility.

3.2 An interpretation of the constructed vNM Utilities

In this section, we make a more general connection between undominatedness and the profile of vNM utilities discussed in Section 3. Indeed, duality theory lends the constructed vNM utility profile $\hat{\mathbf{u}}$ of Theorem 1 a particular kind of interpretation, regardless of whether the candidate allocation $\hat{\mathbf{p}}$ is undominated. We begin by defining the concept of a subgradient that is commonly encountered in convex analysis.

Definition 2 *Let $f : \mathcal{X} \rightarrow \Re$ denote a convex function defined on a convex set $\mathcal{X}$. Let $\hat{x} \in \mathcal{X}$. A vector $\mathbf{v}$ belonging in the ambient space of $\mathcal{X}$ is a subgradient of f at $\hat{x}$ if*

$$f(\hat{x}) + \mathbf{v} \cdot (x - \hat{x}) \leq f(x), \quad \forall x \in \mathcal{X}.$$

[7] For more details on polyhedra and extreme points, the reader is referred to Section 2.2 in [2].

Returning to our model, let $\tilde{\mathbf{p}} \in P$ and define the vector-valued function

$$\mathbf{g} : P \to \Re^{|N^O|m+\sum_{i\in N^C} n_i},$$

such that $\mathbf{g}(\tilde{\mathbf{p}})$ is a column vector whose entries are the first

$$|N^O|m + \sum_{i\in N^C} n_i$$

entries of the right-hand-side of the constraint vector of primal problem (1) applied to an allocation $\tilde{\mathbf{p}}$. That is,

$$\begin{aligned} i \in N^O: \quad \mathbf{g}(\tilde{\mathbf{p}})_{ij} &= \sum_{k \succeq_i j} \tilde{p}_{ik}, \quad j \in M \\ i \in N^C: \quad \mathbf{g}(\tilde{\mathbf{p}})_{il} &= \sum_{j=1}^{m} u_i^l(j)\tilde{p}_{ij}, \quad l \in \{1, 2, ..., n_i\}. \end{aligned}$$

Next, given two vectors $\mathbf{x} \in \Re^{|N^O|m}$ and $\mathbf{w} \in \Re^{\sum_{i\in N^C} n_i}$ we define $\mathbf{u}$ to be a vector-valued function

$$\mathbf{u} : \Re^{|N^O|m+\sum_{i\in N^C} n_i} \to \Re^{nm}$$

where $\mathbf{u}(\mathbf{x}, \mathbf{w})$ consistent with Eq. (4). Echoing Eq. (5) in the proof of Theorem 1, we can rearrange terms and establish the following identity for all $\tilde{\mathbf{p}} \in P$

$$(\mathbf{x}, \mathbf{w})'\mathbf{g}(\tilde{\mathbf{p}}) = \mathbf{u}(\mathbf{x}, \mathbf{w})'\tilde{\mathbf{p}}. \tag{10}$$

We use the primal problem (1) to define an allocation's *efficiency deficit* as the negative of the greatest amount by which it can be dominated by another feasible allocation. Or, equivalently, as the greatest combined ordinal and cardinal efficiency loss that its application can result in. Let $F(\hat{\mathbf{p}})$ denote the feasible region of the primal problem (1) as a function of a candidate allocation $\hat{\mathbf{p}} \in P$. The efficiency deficit of an allocation $\hat{\mathbf{p}}$ is defined as the optimal cost of the primal problem (1) when the allocation appearing in the right-hand-side of the constraints is given by $\hat{\mathbf{p}}$. Formally, it is denoted by a function $D : P \to \Re_-$ such that

$$D(\hat{\mathbf{p}}) = \min_{(\mathbf{p},\mathbf{r},\mathbf{q},\mathbf{s})\in F(\hat{\mathbf{p}})} -\Big(\sum_{i\in N^O}\sum_{j=1}^{m} r_{ij} + \sum_{i\in N^C}\sum_{l=1}^{n_i} q_{il}\Big).$$

Proposition 1 *The efficiency deficit $D(\cdot)$ is a piecewise-linear convex function on the set P.*

Proof. Recall the proof of Theorem 2 and consider the set of extreme points $\mathbf{E}$ of the updated dual feasible region (8). We may write

$$\begin{aligned} D(\hat{\mathbf{p}}) &= \max_{\mathbf{e}\in\mathbf{E}} \Big\{(\mathbf{x^e}, \mathbf{w^e}, \mathbf{y^e}, \mathbf{z^e})'(\mathbf{g}(\hat{\mathbf{p}}), \mathbf{1}, \mathbf{1})\Big\} \\ &= \max_{\mathbf{e}\in\mathbf{E}} \Big\{\mathbf{u}(\mathbf{x^e}, \mathbf{w^e})'\hat{\mathbf{p}} + (\mathbf{y^e}, \mathbf{z^e})'(\mathbf{1}, \mathbf{1})\Big\}. \end{aligned} \tag{11}$$

Since the maximum of a set of linear (and therefore convex) functions is itself convex, the result follows. ■

We are now ready to generalize the insights obtained in the proof of Theorem 1.

Theorem 3 *Consider an allocation* $\hat{\mathbf{p}} \in P$ *and the associated primal LP (1). Suppose the vector* $(\hat{\mathbf{x}}, \hat{\mathbf{w}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ *is an optimal solution of the associated dual LP (2) and consider the vNM utility profile* $\hat{\mathbf{u}}$ *given by Eq. (4). This profile is (a) preference-compatible, and (b) a subgradient of the efficiency deficit* D *at* $\hat{\mathbf{p}}$*.*

Proof. Part (a) follows immediately from dual feasibility. We turn to part (b). The simple argument follows the proof of Theorem 5.2 in Bertsimas and Tsitsiklis [2]. Strong duality implies that

$$(\hat{\mathbf{x}}, \hat{\mathbf{w}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})'(\mathbf{g}(\hat{\mathbf{p}}), \mathbf{1}, \mathbf{1}) = D(\hat{\mathbf{p}}) \overset{(10)}{\Rightarrow} \mathbf{u}(\hat{\mathbf{x}}, \hat{\mathbf{w}})'\hat{\mathbf{p}} + \sum_{i=1}^{n} \hat{y}_i + \sum_{j=1}^{m} \hat{z}_j = D(\hat{\mathbf{p}}).$$

Consider now an arbitrary $\tilde{\mathbf{p}} \in P$. By weak duality (see Theorem 4.3 in [2]), we have

$$\mathbf{u}(\hat{\mathbf{x}}, \hat{\mathbf{w}})'\tilde{\mathbf{p}} + \sum_{i=1}^{n} \hat{y}_i + \sum_{j=1}^{m} \hat{z}_j \leq D(\tilde{\mathbf{p}}).$$

Hence, we may conclude that

$$\mathbf{u}(\hat{\mathbf{x}}, \hat{\mathbf{w}})'(\tilde{\mathbf{p}} - \hat{\mathbf{p}}) \leq D(\tilde{\mathbf{p}}) - D(\hat{\mathbf{p}}), \quad \text{for all } \tilde{\mathbf{p}} \in P.$$

■

4 Directions for Future Research

The results in this paper provide a concise characterization of efficiency in environments with a mix of ordinal and cardinal information on agent preferences. In particular, an allocation is undominated if and only if its efficiency deficit a piecewise-linear convex function on the set of allocations, is zero. We believe that this insight, coupled with the more general optimization framework explored in this work, may prove useful in future research in random-assignment and house-allocation models. In particular, one may frame different kinds of existence questions by setting up a trivial optimization problem (i.e., one with a zero objective), imposing as constraints desired properties of efficiency, equity, and voluntary participation, and examining its dual. A similar approach may be helpful in the comparison of individual allocation mechanisms; in particular, one can attempt to provide bounds on the difference of their ex-ante welfare, for a range of preference-compatible utility profiles.

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