An Empirical Differential Game for Sustainable Forest Management *

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Abstract

We model the role of the world's forests as a major carbon sink and consider the impact that forest depletion has on the accumulation of CO_2 in the atmosphere. Two types of agents are considered: Forest owners who exploit the forest and draw economic revenues in the form of timber and agricultural use of deforested land; and non-forest owners who pollute and suffer the negative externality of having a decreasing forest stock. We retrieve the cooperative solution for this game and show in which cases cooperation allows to partly reduce the negative externality. We analyze when it is jointly profitable to abate emissions, when it is profitable to reduce net deforestation, and when it is optimal to do both things (abate and reduce net deforestation).

Key Words: Game theory, dynamic games, optimal control, deforestation, forest management, emissions, renewable resources.

1 Introduction

World forests cover nearly one third of planet Earth's surface. However, total world forest area is decreasing at an alarming rate. Every year an area equivalent to the size of Costa Rica is deforested (FAO, 2010). World deforestation has become an issue of great international environmental concern for a number of reasons: First, world forests have great ecological value as carbon sinks.

*The second author research is partially supported by MICINN and JCYL under projects ECO2008-01551/ECON, ECO2011-24352, and VA001A10-1, co-financed by FEDER funds. The third author's research is supported by NSERC, Canada.

Second, forests host much of the world's biodiversity. Third, forests protect land and water resources and help prevent land erosion and desertification. In this paper we concentrate mainly on the role of forests as carbon sinks, although the framework used here could be extended to include the other two.

We view forests as a provider of somewhat competing economic and environmental goods. While forest logging brings economic revenues from both timber and agriculture on deforested land in the short run (FAO, 2006), excessive logging can exacerbate the problem of greenhouse gases (GHGs) accumulation in the long run. We have built a model where we account for the accumulation of GHGs in the atmosphere and propose a GHG accumulation dynamics in terms of both anthropogenic emissions and carbon sequestration by the world's forests. The framework used allows to (i) evaluate the impact that forest depletion has on atmospheric GHG accumulation through the so called *reduced-carbon-sequestration effect*; and (ii) compare short-term rewards from high emissions and intensive deforestation policies with its long-term costs due to excessive GHG accumulation and forest depletion.

There exist many papers that deal with the role of excessive GHG accumulation in the atmosphere within a dynamic setting (see, e.g., the early papers by Van der Ploeg and De Zeeuw (1992), Long (1992), Dockner and Long (1993) and the literature review by Jørgensen et al. (2010)). In this literature, emissions are a control variable and the issue is to determine the optimal emissions rate so as to reduce the environmental damage coming from the excessive accumulation of GHGs. Typically, these models concentrate on the difficulty to coordinate on the optimal level of emissions, while treating carbon sequestration as exogenously given.

In this paper, we extend the literature and explicitly account for endogenous carbon sequestration by modelling the role of forests as a carbon sink. Forests are considered as a renewable resource whose evolution has an impact on the accumulation of GHGs in the atmosphere.

There exist a number of papers in the literature that deal with the issue of forest depletion using a dynamic-game approach (e.g., Van Soest and Lensink (2000), Fredj et al. (2004), Fredj et al. (2006), Martín-Herrán and Tidball (2005) and Martín-Herrán et al. (2006)). In these articles, the players are forest owners who exploit the forest to obtain economic revenues, and a donor community, or an environmentally-aware player, that is willing to compensate forest owners who engage in preservation efforts of the resource.

We have merged these two strands of the literature. We have built a dynamic optimization problem where two economic agents interact. On the one hand forest owners exploit (and eventually deplete) the forest. Their actions have an environmental impact on the atmospheric accumulation of GHGs. On the other hand non-forest owners who derive utility from production (i.e., emissions) and disutility from the accumulation of GHGs in the atmosphere. In this setting, it is this disutility that they experience that, in some cases, may turn them into donors who seek for forest conservation.

In our model, forest owners have an incentive to deforest since deforestation increases their economic revenues. Conversely, non-forest owners have an interest to preserve forests for their value as a carbon sink. This modelling framework allows to capture both the high opportunity cost to reduce deforestation and the negative economic externality that forest owners inflict on non-forest owners as a consequence of their deforestation policy. Unlike the other papers aforementioned we do not focus solely on forest conservation but also on its impact on GHG accumulation. Non-forest owners are to decide what is their optimal level of emissions. The parameters of the model have been calibrated to fit real data. The jointly optimal outcomes are compared with non-cooperative or business-as-usual policies. We show that cooperation allows to partly reduce the negative externality and analyze when it is profitable to abate emissions, when it is profitable to reduce net deforestation; and when it is optimal to do both things (abate and reduce net deforestation).

The remainder of the paper is organized as follows: in Section 2, we present the model and the economic problem for the two types of agents. In Section 3, we characterize analytically the non-cooperative optimal policies for each player. In Section 4, we compute the cooperative optimal policies, and compare them to their non-cooperative counterparts. We also perform a sensitivity analysis. Our results are summarized in Section 5.

2 The model

We consider two types of agents: forest owners and non-forest owners. Forest owners are modeled as environmentally unconcerned agents who only care about the forest revenues obtained with deforestation. We suppose forest owners to neglect the environmental impacts of their actions, i.e., they do not consider the consequences that their deforestation policy brings out in terms of GHG accumulation. On the other hand, non-forest owners get revenues from the production of economic goods. Their productive activity generates emissions and non-forest owners do take into account the negative effects of current emissions policies on the accumulation of GHGs in the atmosphere. This way of modelling allows to capture the negative externality that forest owners create on non-forest owners through the so called *reduced-carbon-sequestration effect*.

We present the objectives of the two players. In what follows we use subscript FO to denote *forest owners* and subscript NF to denote *non-forest owners*.

2.1 The problem of forest owners

Forest owners maximize their discounted stream of net revenues. Forest revenues depend on their afforestation and deforestation rates $A(t)$ and $D(t)$, respectively, as well as on the existing forest area $F(t)$ measured in hectares. Net revenues are discounted at rate r_{FO} throughout a fixed and finite time horizon, given by time T. The rate r_{FO} can be viewed as an intertemporal rate of substitution. Net revenues include gross revenues $R(t)$, afforestation costs $\kappa_1 A(t)$ and deforestation costs $\kappa_2 D(t)$, where κ_1 and κ_2 are respectively the per-hectare afforestation and deforestation costs. The objective of forest owners is the following:

$$\max_{A(t),D(t)} \int_0^T e^{-r_{FO}t} \left[R(t) - \kappa_1 A(t) - \kappa_2 D(t)\right] dt, \tag{1}$$

where $A(t) \in [0, A_{\max}]$ and $D(t) \in [0, D_{\max}]$. The upper bounds for afforestation ($A_{\max}$) and deforestation ($D_{\max}$), reflect the idea that there is a physical limit in the short term to afforestation and that deforestation is subject to some regulation that allows for it within some limits. The value of $D_{\max}$ is set to fit the observed deforestation world figures provided by FAO (2006). The definitions of all parameters, their values and their sources are provided in Appendix A.

We assume that the evolution over time of the forest area can be well approximated by the following linear-differential equation:

$$\dot{F}(t) = A(t) + \eta F(t) - D(t), \quad F(0) = F_0, \tag{2}$$

where η is a positive parameter, and F_0 is the initial world's forest area in 2005 (FAO, 2006) and equals nearly four billion hectares. Equation (2) is an extension of Van Soest and Lensink (2000) and Fredj et al. (2006), where $A = \eta = 0$ in the first and $A = 0$ in the second. The specification used in (2) is linear for simplicity and approximates reasonably well forest expansion within a large interval around current world forest area $F(0)$. The linear form of the dynamics will enable us to obtain analytical solutions.

Forest owners obtain revenues from selling timber and agriculture products. Denote by $q(t)$ the quantity of timber put on the market at time t, and let the price $p(t)$ be given by the following linear-inverse demand:

$$p(t) = \overline{p} - \theta q(t), \tag{3}$$

where $\overline{p}$ is the choke price that makes demand equal to zero, and θ is the average price elasticity of demand. The values of parameters $\overline{p}$ and θ have been calibrated using data given by FAO on timber prices and quantities.

The quantity $q(t)$ comes from two different sources, namely, clear felling and selective logging, and is given by

$$q(t) = nD(t) + n\gamma\delta F(t), \tag{4}$$

where $nD(t)$ is the amount of wood retrieved from clear-felling an area $D(t)$ and the product $n\gamma\delta F(t)$ stands for total selective-logging yield which is lower (in per-hectare terms) than the one obtained through clear felling. Parameter n denotes the per-hectare timber yield and is typically measured in stems per hectare or cubic meters of timber per hectare. FAO (2006) provides an estimate for this parameter. Clear felling an area $D(t)$ reduces total forest size by the same amount. However, unlike deforestation, selective logging is assumed here to have no impact on total forest land. "[Selective logging]*...is not necessarily destructive and can be done with low impact on the remaining forests, if the proper techniques are applied*".[1] Clearly, for selective logging to have a

[1]Source: http://www.fao.org/forestry/news/48681/en/

negligible environmental impact, its per hectare yield per unit of area must be much lower with respect to clear felling. This lower yield is accounted for by parameter γ ($\gamma << 1$). Finally, according to FAO (2006), roughly one third of the world's forests are used primarily for the production of wood and non-wood forest products. Parameter δ takes into account the fact that only a fraction of the world's forests are actually being exploited.[2]

Agriculture revenues are equal to prices times yields of the different crops grown. For simplicity, we suppose that forest owners grow a single agricultural good that we model as a composite good made of four representative crops that are commonly related to deforestation processes. This good is sold in international markets at a given price p_A.[3] The total yield at time t depends on the size of available (deforested) land, given by $\bar{F}-F(t)$, where $\bar{F}$ stands for the maximum size or carrying capacity of the forest, and on the soil productivity $x(t)$. As in Andrés-Domenech et al. (2011) -see also Van Soest and Lensink (2000) for a simpler version- we model $x(t)$ as follows:

$$x(t) = \overline{x} + \alpha(t)D(t) - \beta\frac{\bar{F} - F(t)}{\bar{F}}. \tag{5}$$

The above expression of total productivity of land $x(t)$ is the sum of three terms. The first one is a constant productivity term $\overline{x}$ that measures the average yield in tons of crop per hectare of land of a representative agricultural good. The second term, $\alpha(t)D(t)$, captures the idea that newly deforested land $D(t)$ is more productive. Variable $\alpha(t)$ measures the increase in *total* average per-hectare production resulting from deforesting an area $D(t)$. The third term, $-\beta\frac{\bar{F}-F(t)}{\bar{F}}$, accounts for the positive externality that forests generate on nearby agricultural land. Forests are seen as a source of rain and a protective element to agricultural land. Parameter β measures the decrease (increase) in soil quality, and therefore in agricultural productivity, caused by forest depletion (expansion). The productivity increase of newly deforested land is given by:

$$\alpha(t) = \frac{\psi\overline{x}}{\bar{F} - F(t)}. \tag{6}$$

Newly deforested land is more productive and parameter ψ measures the factor by which productivity is increased. However, this extra productivity needs to be normalised among all agricultural land. We divide the extra yield, $\psi\overline{x}$, by total agricultural surface area, $\bar{F} - F(t)$, otherwise the term $\alpha(t)D(t)$ in equation (5) would overestimate the real impact that deforesting an area $D(t)$ has.[4] Note

[2] The equation that we have used for q is a small variation of the one presented by Van Soest and Lensink (2000). In their case γ and δ are assumed equal to one. We follow here the more comprehensive specification used by Andrés-Domenech et al. (2011).

[3] The price p_A is constant, unlike $p(t)$, due to the fact that agricultural production in deforested land represents only fraction of world total agricultural land.

[4] Agricultural revenues are obtained by multiplying productivity (5) by total agricultural land. Hence, equation (5) has to account for average *per-hectare* productivity measured in tons of crop per hectare. For this reason, the term $\alpha(t)D(t)$ cannot be understood as the extra productivity of newly deforested land, but rather as the normalised productivity increase that *newly* deforested land has on *total* agricultural land.

that Van Soest and Lensink (2000) assume a constant productivity increase, i.e., $\alpha(t) \equiv \alpha$.

Putting together revenues from timber sales and agricultural products, we get the following expression for gross revenue:

$$R(t) = p(t)q(t) + p_A x(t) \left(\bar{F} - F(t)\right). \tag{7}$$

To recapitulate, forest owners maximize their net discounted economic revenues with respect to their deforestation and afforestation efforts $D(t)$ and $A(t)$, respectively (problem (1)), subject to the forest dynamics in (2).

2.2 The problem of non-forest owners

The economic problem of non-forest owners is quite different. Non-forest owners optimize a two-part objective function. The first part consists of a short-run gain that non-forest owners derive from producing and consuming economic goods. The production of these goods generates pollution as a by product and this pollution affects their utility. For simplicity, we have supposed here that the carbon intensity of the economy is constant. Hence, *ceteris paribus*, producing more goods is equivalent to emitting more.[5] Denote by $E(t)$ the GHGs emissions by non-forest owners; and by the concave increasing function $G(E)$ the payoff generated in terms of goods production. We adopt the following functional form:

$$G(E(t)) = aE(t) - \frac{1}{2}bE^2(t), \tag{8}$$

where parameters a and b are positive and have been fixed in order to ensure that $G'(E) > 0$ for the relevant range of emissions. This specification is similar to the one proposed in, e.g., Dockner and Long (1993) and Breton et al. (2005) with the only difference that we have included parameter b to calibrate $G(E(t))$ at current GDP at the world level.

The second term in the objective of non-forest owners represents an economic loss or damage related to the accumulation of emissions in the atmosphere. Denote by $S(t)$ the instantaneous stock of GHGs (e.g., stock of CO_2) in the atmosphere at a given time t. According to the IPCC (2007) increases in the atmospheric concentration of GHGs result in sea water level rising, temperatures increasing and sea water acidification. These processes are all related to economic and environmental damages. We assume that the damage cost is given by a convex increasing function $L(S)$, with $L''(S) > 0$. Although we acknowledge the existence of thresholds, extreme events and jumps in the damages,[6] our formulation, which is very common in the literature (see, e.g., Benchekroun

[5] One could think of a more refined formulation where the carbon intensity of the economy can adjust and production increases can be compatible with constant levels of emissions or even with emission decreases.

[6] For instance, a small increase in the atmospheric concentration of GHGs can bring a quantitatively different damage, and large increases may trigger qualitatively different damages (e.g., massive ice cap melting, dissolution of coral reefs as a result of extreme oceanic acidification, etc).

and Long, 2002; Dockner and Long, 1993; Van der Ploeg and De Zeeuw, 1992; Breton et al., 2006), smooths the impacts of such phenomena rather than dealing with them explicitly. Needless to say, accounting properly for non linearities and threshold effects in the damage cost would lead to a model of much greater complexity.

This being said, for a specific function to qualify as a good candidate to model such damages we can think of yet another necessary requirement: Greenhouse gases, and most particularly CO_2, have always been present in the atmosphere and represent a basic element to the existence and development of life (e.g., plants). It is clear that more than the existence it is the *excessive* accumulation of atmospheric GHGs that poses the problem. We adopt the following specification of $L(S)$ that captures in a simple way all these elements:

$$L(S(t)) = c(S(t) - \underline{S})^2, \tag{9}$$

where $\underline{S}$ is a natural threshold, beyond which economic and environmental damages are considered to be excessive. In practical terms, choosing a reasonable value for $\underline{S}$ -given the specification above- amounts to choosing a level of atmospheric GHGs for which there is no perceived damage. We identify $\underline{S}$ with the pre-industrial level of GHGs (see, e.g., Bahn et al. (2008)).

Taking into account the gain function $G(E)$ and the damage function $L(S)$, we obtain the objective functional that non-forest owners maximize:

$$\int_0^T e^{-r_{NF} t} \left[G(E(t)) - L(S(t))\right] dt - \phi\left(S(T)\right) e^{-r_{NF} T}, \tag{10}$$

where r_{NF} is the intertemporal rate of substitution, and $\phi(S(T))$ is a salvage value. Note that for the sake of generality, we do not require that both players discount their stream of payoffs at the same rate, i.e., r_{NF} need not be equal to r_{FO}.

Non-forest owners are modelled as forward looking agents who consider the long-term impact of their decisions. The stock of emissions accumulates slowly and then has a long-term impact on non-forest owners' payoffs. Therefore, it is sensible to have a scrap value function somehow related to the stock of emissions at the terminal date of the planning horizon. Such scrap value function can be generically written as $\phi(S(T))$. It is reasonable to think that whatever the GHG stock at the terminal date, it will strongly impact future payoffs due to the long-term persistence of greenhouse gases in the atmosphere. One could think of a more sophisticated scrap value function that also depends on the final forest stock or on the emissions policy followed after the terminal date. Or even define the scrap value function as an identical problem to the one presented above in equation (10). Because we want to keep the problem as simple as possible, and because we want to be able to say something that is irrespective of what policies are chosen after the terminal date, we have chosen a formulation for $\phi(S(T))$ that depends on the terminal stock of greenhouse gases alone:

$$\phi\left(S(T)\right) = \int_T^{2T} e^{-r_{NF}(s-T)} L(S(T))\, ds. \tag{11}$$

Although the salvage function in (11) is simple, it satisfies the following intuitive requirements: (i) It reflects the idea that the terminal stock of GHGs matters and has an impact on future payoffs; (ii) it is easy to compute and does not depend on (potentially) unknown future policies; (iii) it keeps discounting in a natural way the cost of future environmental damages; and (iv) the time span considered for the scrap value function is related to the planning horizon. In fact, if the planning horizon chosen is short, then the weight given for future environmental damages will likely be small as well and vice versa.

Non-forest owners maximize their payoffs in (10) by adjusting their emissions, and their decision has an impact on the state of the system. Emissions, in our model, are supposed exclusively anthropogenic and are given entirely by non-forest owners' emissions. By this we do not mean that forest owners do not emit but rather that their contribution to global emissions is negligible. The dynamics of the emissions rate $E(t)$ is then given by:

$$\dot{E}(t) = V(t)E(t), \quad E(t) \geq 0, \; E(0) = E_0. \tag{12}$$

The dynamics of emissions in equation (12) can also be written in a more familiar way:

$$\frac{\dot{E}(t)}{E(t)} = V(t),$$

where $V(t)$ denotes the instantaneous speed of variation of emissions. For the sake of realism $V(t)$ has been modelled as a bounded control variable (i.e., $V_{\min} \leq V(t) \leq V_{\max}$), with $V_{\min} < 0$ and $V_{\max} > 0$. In the literature, it is more common to see emissions as a flow variable. As in Andrés-Domenech et al. (2011), we treat emissions as a stock and the speed of variation of emissions as a bounded control variable. This way of modelling allows to better account for the inertia of the productive and economic system. Indeed, emissions take time to adjust and the upper and lower bounds on $V(t)$ simply reflect this idea that emissions cannot be increased or decreased at whatever rate. One can think of these bounds as being given by the existence of technical, economic and/or political constraints.

The evolution of the stock of greenhouse gases in the atmosphere depends on emissions and carbon sequestration by world forests and oceans. World forests sequester carbon as they grow and according to IPCC (2000) and FAO (2006) approximately half of the dry weight of forest biomass is carbon. To model carbon sequestration by forests one could measure the variation of total forest biomass. However, this would present two main difficulties. First, the variation in total carbon biomass is difficult to measure. And second, measuring carbon sequestration through the variation in forest biomass underestimates total carbon sequestration since timber captures are neglected. To overcome this problem, we have made the simplifying assumption that forest owners manage a representative forest whose trees grow -volume wise- at an average and constant rate. Having a representative forest whose growth rate is constant allows expressing carbon sequestration as a linear function of forest area alone (i.e., carbon sequestered per hectare of forest land and per unit of time). The advantage of having carbon sequestration in terms of forest area -rather than in terms of biomass variation- is that one can easily consider timber captures, while gaining a tractable and understandable way to measure carbon sequestration.

Further, note that by measuring carbon sequestration as a function of forests' surface area, one can account for the so called *reduced-carbon sequestration effect* that is based on the simple principle that a tree that is cut cannot grow (i.e., cannot sequester carbon). Thus, it is straightforward to see that deforestation has a negative impact on carbon sequestration due to the reduction in forest area that it induces. Expression (13) below captures the dynamics of the atmospheric concentration of carbon in terms of the forest stock, where parameter φ reflects the amount of carbon sequestered per hectare of forest and per unit of time.

We have considered a second carbon sink, the oceans. Denote by W the amount of carbon that they sequestrate. The carbon uptake by the oceans has remained relatively stable during the last few years, for this reason W has been assumed constant for simplicity even if there exist small year-to-year variations due to el Niño effects (Le Quéré et al. (2009)). The evolution of the stock of pollution is, then, given by the following differential equation:

$$\dot{S}(t) = E(t) - \varphi F(t) - W, \quad S(t) \geq 0, \quad S(0) = S_0. \tag{13}$$

To wrap up, non-forest owners maximize their payoff in (10) adjusting the instantaneous variation of emissions $V(t)$, subject to equations (12) and (13) and given the fact that the solution to (2) is inherited from forest owners' problem.

3 Individual optimization

In this section we characterize the optimal strategies of the two players when they act independently. As forest owners' payoffs do not depend on emissions or GHG accumulation, their payoffs are independent of the action of non-forest owners. On the other hand, non-forest owners' payoffs are affected by forest owners' actions through the evolution of the forest stock. In this setting, where there is a one-way interaction, Nash and Stackelberg equilibria coincide. Further, open-loop and feedback information structures yield the same result. Given this, we can first solve the economic problem of forest owners, and next optimize for non-forest owners taking the evolution in the forest stock as given.

3.1 Forest owners

Forest owners maximize their revenues in (1) subject to (2)-(7). The following proposition provides the optimal solution to their control problem.

Proposition 1 *For the parameter domain defined in Appendix A the optimal* *control, state and co-state variables are given by:*[7]

$$A^*(t) = 0, \quad D^*(t) = D_{\max} \ \textit{for all}\, t \in [0,T],$$

$$F(t) = \left(F_0 - \frac{D_{\max}}{\eta}\right) e^{\eta t} + \frac{D_{\max}}{\eta}, \tag{14}$$

$$\lambda(t) = \frac{1}{\eta - r_{FO}} \left[1 - e^{(\eta - r_{FO})(T-t)}\right] \big\{(2\theta n D_{\max} - \overline{P}) n\gamma\delta$$

$$+ p_A(\overline{x} - 2\beta) + 2F(t)\left[\theta n^2\gamma^2\delta^2 + p_A\beta\frac{1}{\overline{F}}\right]\bigg\}. \tag{15}$$

Proof. See Appendix B. ■

The results show that forest-owners' optimal strategy consists in deforesting at maximum admissible level and not afforesting at all. As the problem is linear in the afforestation effort, and afforestation is a pure cost in our setting, then the optimal strategy is obviously to set $A(t)$ at its lowest admissible value, i.e., $A(t) = 0$. Further, the marginal revenue from agricultural activity is positive for all admissible values of $D(t)$, including $D_{\max}$. Therefore, there is an incentive to deforest at maximum level. These results follow from the fact that, for our parameter domain, we have $\lambda(t) \leq 0$, for all t. Indeed: (i) the term $\frac{1}{\eta - r_{FO}}\left[1 - e^{(\eta - r_{FO})(T-t)}\right]$ is always negative since $\frac{1}{\eta - r_{FO}}$ and $\left[1 - e^{(\eta - r_{FO})(T-t)}\right]$ are of opposite sign regardless of the value of η and r_{FO}; and (ii) $(2\theta n D_{\max} - \overline{P})n\gamma\delta + p_A(\overline{x} - 2\beta) > 0$, for all admissible values of $D(t)$, including $D_{\max}$. Deforestation is mainly driven by the revenues obtained from growing agricultural products on deforested land, rather than by the timber revenues that arise from deforestation itself. This is in line with other studies, e.g., Barbier and Rauscher (1994), Barbier and Burgess (2001) and FAO (2006), which suggested that deforestation for agricultural purposes is the main explanatory factor of forest depletion worldwide.

3.2 Non-forest owners

Non-forest owners maximize their payoff given by (10) and take into account the values of the three state variables, namely forest area, F, emissions, E, and the stock of accumulated emissions in the atmosphere, S. The optimal solution depends on the length of the planning horizon and on the intertemporal discount rate. For the values of our parameters, the solution is constant ($V^* = V_{\max}$) as long as the planning horizon (T) is less than approximately forty years, (i.e., $T \lesssim 40$).[8] The following proposition provides the optimal solution to the problem of non-forest owners and the optimal time paths for control and state variables in such case.

[7]The second-order sufficient optimality conditions are satisfied for this and all the problems studied in this paper.

[8]The determination of the exact planning horizon beyond which Proposition 2 does not hold depends on the intertemporal discount rate. As we will see for every value of the discount rate we can obtain the maximum value of T for which Proposition 2 holds.

Proposition 2 *For the parameter domain defined in Appendix A and $T \lesssim 40$, the optimal control and state variables are given by:*

$$V(t) = V^* = V_{\max} \text{ for all } t \in [0,T],$$
$$E(t) = E_0 e^{V^* t}, \tag{16}$$
$$S(t) = S_0 - \frac{\varphi}{\eta} t D_{\max} - \frac{E_0}{V^*}(1 - e^{V^* t}) + \frac{\varphi}{\eta}\left(F_0 - \frac{D_{\max}}{\eta}\right)(1 - e^{\eta t}). \tag{17}$$

Proof. See Appendix C. ∎

As pointed out above, the optimal control V^* depends on the planning horizon considered. For a relatively short horizon, i.e., $T \lesssim 40$, the optimal solution is constant of the type $V^* = V_{\max}$ all along. The solutions showed in Proposition 2 hold for as long as there is no switching time. For longer horizons, i.e., $T \gtrsim 40$, the optimal solution is to apply the control $V = V_{\max}$ for some time and then switch to a cleaner regime. For much longer horizons (i.e., $T > 100$), it is possible that the optimal solution consists of switching not once but several times. In all cases, the different switching times and the number of switches depend on the adopted value for T. Denote by $\tilde{t}_V$ the optimal switching time. Then the optimal solution for $40 \lesssim T \leq 100$ can be summarized as follows:

$$V(t) = \begin{cases} V_{\max}, & \text{for } t \leq \tilde{t}_V \\ V_{\min}, & \text{for } t > \tilde{t}_V. \end{cases}$$

In Appendix D we have solved the problem for the case where there is only one switching time, and characterized the first-order conditions that apply in that case. Retrieving the actual switching time, however, represents a challenge. This is mainly due to the change in the evolution of the state and co-state variables as a consequence of changes in the switching time itself. The first-order conditions before and after the switch will only be satisfied if the exact switching time is chosen. This poses a problem in determining the actual switching time since one has to try with infinitely many possibilities and the first-order conditions will only be satisfied if the exact one is chosen.

To overcome this problem we have developed an algorithm to obtain the optimal switching time that is approximated to the integer time at which it is best to switch. The algorithm proposed consists on evaluating the sum of payoffs for all the possible scenarios (i.e., all the possible switching times). Among them, we then select the integer time for which shifting regime (from $V_{\max}$ to $V_{\min}$) yields greater payoffs. A sketch of the algorithm can be found in Table 1.

Suppose for instance that our planning horizon and discount rate were fixed at, e.g., $T = 50$, $r_{NF} = 0.02$. Figure 1 gives the payoffs of non-forest owners in the y-axis for each possible switching time (x-axis). We observe, for this particular case, that switching from $V_{\max}$ to $V_{\min}$ after $\tilde{t_V}$ (where $\tilde{t_V} = 17$ years) is the best thing to do.

We can generalize the algorithm presented in Table 1 and let the planning horizon T vary while keeping the discount rate r_{NF} constant. So doing we obtain the best switching time for each different planning horizon.

Table 1: Sketch of algorithm used to compute the optimal switching time $\tilde{t}_v$

Fix the length of the planning horizon (T) and the discount rate (r_{NF})
for all possible integer switching times (t_v) **do**
  Payoff(t_v) = Discounted sum of payoffs before switch t_v
    + Discounted sum of payoffs after switch t_v
    + Scrap value function
end
Select the t_v whose Payoff is greater

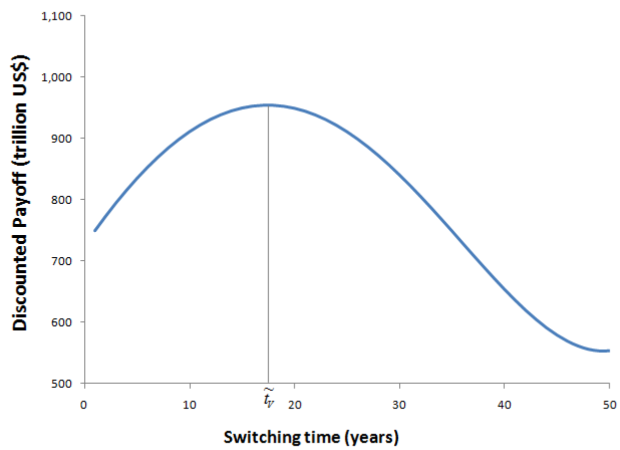

Figure 1: Payoffs as a function of the switching time

Figure 2 gives the optimal switching time for each possible planning horizon T. To better understand this figure it is important to distinguish the difference between three concepts. First, parameter T denotes the planning horizon of the problem. Second, the value T^s denotes the threshold planning horizon, that is, the minimum planning horizon beyond which there is a switch. And finally, since the switching time does not coincide with T^s; the value $\tilde{t_V}$ denotes the time in which the switch actually occurs.

The 45-degree diagonal indicates that no switch is applicable. The shortest planning horizon for which there is a switch - T^s- is the first element of the curve off the diagonal. Figure 2 illustrates the fact that it pays to emit more in the short run. It also shows that for longer planning horizons it is comparatively less attractive to apply $V_{\max}$. This result is related to the existence of the non-linear damage function $L(S)$, by which the environmental damage increases when GHGs accumulate due to excessive emissions.

In Figure 2, $T^s = 38$. This means that there is no switch if the planning horizon is *short* (i.e., less than 38 years). Conversely, there is a switch if $T \geq T^s$. As mentioned before, T^s and $\tilde{t_V}$ do not coincide, not even when $T = T^s$. Put

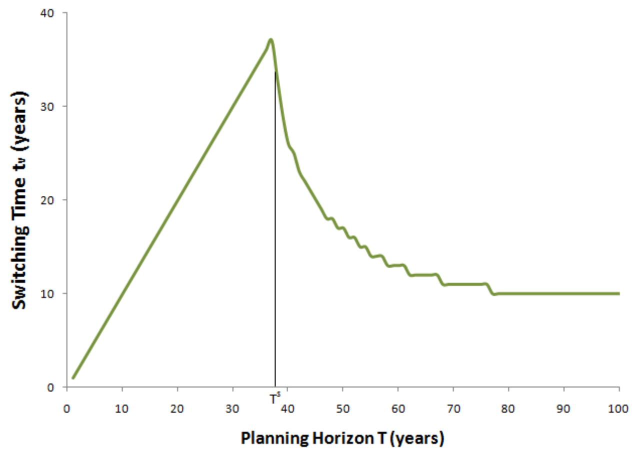

Figure 2: Optimal switching time for every planning horizon

differently, if the planning horizon is long enough non-forest owners recognize the need to switch to a cleaner regime, but the switch will take place some time before the terminal date. Note that the pair $\left(T = 50,\ \tilde{t_V} = 17\right)$ that we previously obtained in Figure 1 is now just one point of the curve displayed in Figure 2.

We can further generalize our algorithm for any value of r_{NF}. In the previous two figures, r_{NF} was set equal to 0.02 (2%). The previous results are compared with two other alternative scenarios $r_{NF} = 1\%$ and $r_{NF} = 3\%$ in Figure 3.

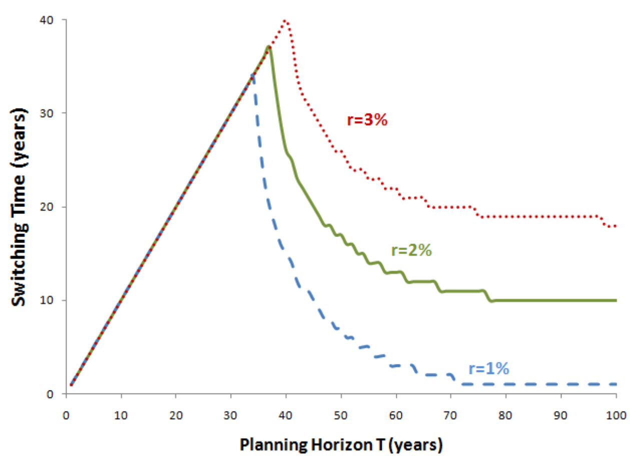

Figure 3: Impact of the discount rate on the switching time

From Figure 3 one can obtain a double message: First, when the discount rate is lower, non-forest owners internalize earlier the negative externality coming from the accumulation of GHGs in the atmosphere. This can be inferred from the fact that T^s is lower for lower discount rates. In particular we have that $T^s = 35$ if $r = 1\%$; $T^s = 38$ if $r = 2\%$; and $T^s = 41$ if $r = 3\%$. Second, the longer the time horizon used, the earlier the switch, i.e., the three curves are downward slopped.

To summarize, it is optimal for non-forest owners to increase emissions if $T \lesssim 40$. If $T \gtrsim 40$, it will be better to switch to a cleaner regime ($V = V_{\min}$) at some time $\tilde{t_V}$. The optimal time of the switch depends directly on the planning horizon and the discount rate used. A simple folk conjecture says that the longer the planning horizon and/or the smaller the intertemporal discount rate, the sooner this switch will arrive. This is related to the existence of the damage function $L(S)$ that yields greater (cumulative) losses for lower discount rates and longer planning horizons.

It has been showed how to determine the switching time. To put things into perspective one can compare in Figure 4 the difference in payoffs between the payoff with the optimal solution with switching time, $\pi(\widehat{V})$, versus the payoffs $\pi(V_{\max})$ and $\pi(V_{\min})$ obtained by applying the constant (and sub-optimal) solutions: $V = V_{\max} \forall t \in [0, T]$ and $V = V_{\min} \forall t \in [0, T]$ respectively.

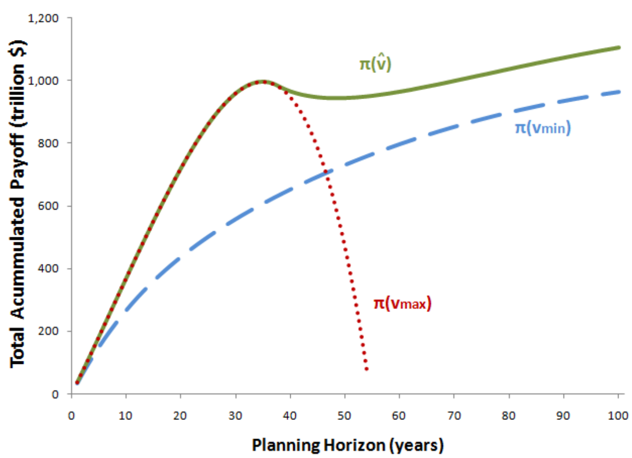

Figure 4: Comparing $\widehat{V}$ with $V_{\min}$ and $V_{\max}$

The value of $\pi(\widehat{V})$ in Figure 4 is obtained by computing expression (10) for $r_{NF} = 2\%$ along the optimal path for $E(t)$ and $S(t)$. For $T < 38$, $\pi(\widehat{V})$ and $\pi(V_{\max})$ coincide. If $T \geq 38$ the curve $\pi(\widehat{V})$ is obtained by applying a switch.

So far we have analyzed the optimal emissions policy. It is also important to analyse the sign of the shadow price of the forest stock, λ_F. This shadow price is positive regardless of the time horizon and discount rate considered.

The positive sign of the co-state λ_F is directly related to the ability that forests have to sequester carbon. Since the increase in forest area is directly related to the enhancement of carbon sequestration (see expression (13)); then, whatever the value of F and S,[9] increasing marginally the forest area implies marginal reductions in S, meaning smaller environmental losses (see expression (9)). This is a qualitative aspect.

At the same time, we have seen that the importance of reducing emissions is directly related to the length of the planning horizon and inversely related to the discount rate. Likewise, the marginal value that non-forest owners attach to an additional hectare of forest is greater when the planning horizon is longer and the discount rate is lower. This is a more quantitative aspect.

In short, unlike forest owners, non-forest owners are interested in increasing total forest area and this is reflected by the sign of λ_F. If we compare the different way in which forest owners and non-forest owners evaluate an additional hectare of forest, it is clear that there exists an environmental externality. As we have seen, forests have at least two uses: (i) the provision of economic revenues; and (ii) carbon sequestration. These uses are competing and somewhat excluding. Forest owners create a negative externality on non-forest owners with their net deforestation policy. Hence, the question is: Should this negative externality be corrected?

Given the existing property rights over the forest, and the fact that forest owners' payoffs are a decreasing function of total forest area, reducing net deforestation is harmful for forest owners. Therefore, the answer to this question depends on whether an additional unit of forest can generate an increase in the payoff of non-forest owners, such that it more than compensates the reduction in forest owners' revenues when they apply a more environmentally friendly deforestation/afforestation policy. If that is the case, then it will be jointly optimal to correct the externality, or at least part of it. In the next section we compute joint payoffs to give an answer to the question raised above. We also compare the cooperative scenario to the *status-quo* individual equilibrium results.

4 Cooperative solution

In the previous section we determined the non-cooperative (*status-quo*) strategies for both forest owners and non-forest owners. We saw that forest owners find it optimal to deforest as much as possible and to not afforest. On the other hand, non-forest owners suffer a negative environmental externality coming from the depletion of the forest via the *reduced-carbon-sequestration effect* that states the simple idea that a tree that is cut cannot grow and thus cannot sequester carbon. This has an impact in the concentration of GHGs in the atmosphere and leads to payoff losses to non-forest owners.

A relevant question to address is whether cooperation can improve welfare. The jointly optimal solution can be obtained by jointly optimizing the payoff functionals of the two players. To obtain this solution we suppose that forest

[9] Clearly we are referring here to values of S above $\underline{S}$.

and non-forest owners adopt the same discount rate r. This assumption is made to avoid giving implicitly different weights to players' streams of payoffs. Note that dealing with the general case of two different discount rates complicates the computations, but does not cause any conceptual difficulty.

The joint-optimization problem is as follows:

$$\max_{\substack{0\leq A(t)\leq A_{\max},\\ 0\leq D(t)\leq D_{\max},\\ V_{\min}\leq V(t)\leq V_{\max}}} \int_0^T e^{-rt}\left[R(F(t),D(t))+G(E(t))-L(S(t))\right]dt-\phi(S(T))e^{-rT}$$

$$\begin{aligned}\text{s.t.:}\quad & \dot{F}(t)=A(t)+\eta F(t)-D(t), \quad \bar{F}\geq F(t)\geq 0,\ F(0)=F_0,\\ & \dot{E}(t)=V(t)E(t), \quad E(t)\geq 0,\ E(0)=E_0,\\ & \dot{S}(t)=E(t)-\varphi F(t)-W, \quad S(t)\geq 0,\ S(0)=S_0,\end{aligned}$$

where A, D and V are the three control variables. The joint payoff is maximized subject to the dynamics of the forest area, emissions, and stock of greenhouse gases in the atmosphere.

The Hamiltonian of the cooperative problem is:

$$\begin{aligned}H^c\left(F,E,S,A,D,V,\lambda_F^c,\lambda_S^c,\lambda_E^c\right) &= R(F,D)+G(E)-L(S)+\lambda_F^c[A+\eta F-D]\\ &\quad+\lambda_S^c[E-\varphi F-W]+\lambda_E^c VE,\end{aligned}$$

where λ_F^c, λ_E^c, λ_S^c denote the co-state variables associated with the forest stock, emissions and the stock of GHGs respectively. All the variables with a superscript c refer to cooperation as opposed to the non-cooperative outcomes retrieved before.

The Lagrangian of the cooperative problem can be written as:

$$\begin{aligned}\mathcal{L}^c\left(F,E,S,A,D,V,\lambda_F^c,\lambda_S^c,\lambda_E^c,w_1^c,w_2^c\right) &= H^c\left(F,E,S,A,D,V,\lambda_F^c,\lambda_S^c,\lambda_E^c\right)\\ &\quad+w_1^c D+w_2^c\left(D_{\max}-D\right),\end{aligned}$$

where $w_1^c(t)$ and $w_2^c(t)$ are the Lagrangian multipliers associated with the deforestation rate.

The first-order optimality conditions read:

$$\max_{\substack{0\le A\le A_{\max},\\ 0\le D\le D_{\max},\\ V_{\min}\le V\le V_{\max}}} \mathcal{L}^c\left(F,E,S,A,D,V,\lambda_F^c,\lambda_E^c,\lambda_S^c,w_1^c,w_2^c\right), \tag{18}$$

$$\dot{F} = A+\eta F-D, \quad \bar{F}\ge F\ge 0,\ F(0)=F_0, \tag{19}$$

$$\dot{E} = VE, \quad E\ge 0,\ E(0)=E_0,\ V\in[V_{\min},V_{\max}],$$

$$\dot{S} = E-\varphi F-W, \quad S\ge 0,\ S(0)=S_0,$$

$$\dot{\lambda}_F^c = r\lambda_F^c - \frac{\partial\mathcal{L}^c}{\partial F}, \quad \lambda_F^c(T)=0, \tag{20}$$

$$\dot{\lambda}_S^c = r\lambda_S^c - \frac{\partial\mathcal{L}^c}{\partial S}, \quad \lambda_S^c(T)=-\frac{d\phi(S(T))}{dS(T)}, \tag{21}$$

$$\dot{\lambda}_E^c = r\lambda_E^c - \frac{\partial\mathcal{L}^c}{\partial E}, \quad \lambda_E^c(T)=0, \tag{22}$$

$$w_1^c\ge 0,\ w_1^cD=0,\ D\ge 0,$$

$$w_2^c\ge 0,\ w_2^c(D_{\max}-D)=0,\ D_{\max}\ge D.$$

The necessary condition for the maximization problem in (18) with respect to the deforestation rate reads:

$$\frac{\partial\mathcal{L}^c}{\partial D}=0; \quad -2\theta n^2\left[\gamma\delta F+D\right]+\overline{p}n+p_A\psi\bar{Z}+w_1-w_2-\kappa_2-\lambda_F^c=0. \tag{23}$$

Because the Lagrangian function is linear in the afforestation rate, A, and $\frac{\partial\mathcal{L}^c}{\partial A}=-\kappa_1+\lambda_F^c$, the optimal afforestation rate is a bang-bang policy as follows:

$$A(t)=\begin{cases} 0 & \text{if } -\kappa_1+\lambda_F^c(t)<0,\\ \widetilde{A}\in[0,A_{\max}] & \text{if } -\kappa_1+\lambda_F^c(t)=0,\\ A_{\max} & \text{if } -\kappa_1+\lambda_F^c(t)>0. \end{cases} \tag{24}$$

Just as before, λ_F^c appears in the optimality conditions for A and D. The only change with respect to the non-cooperative solution is that now λ_F^c captures the negative valuation of an extra hectare of forest (forest owners) as well as the positive effect that increasing forest area has on carbon sequestration (non-forest owners). Hence, now, λ_F^c can take either positive or negative values depending on which effect dominates. Furthermore, unlike in the non-cooperative case where the sign of λ_F^c was constant along the planning horizon for both players; now, nothing prevents that the sign of λ_F^c changes as time evolves. Hence, it is possible that we have a switch in either the afforestation rate, or the deforestation rate, or both throughout the planning horizon.

The differential equation (20) for the costate variable reads:

$$\dot{\lambda}_F^c = (r-\eta)\lambda_F^c+2\theta n^2\gamma\delta(D+\gamma\delta F)+2p_A\beta\frac{F}{\overline{F}}+p_A(\overline{x}-2\beta)-\overline{p}n\gamma\delta+\varphi\lambda_S^c,$$

$$\lambda_F^c(T)=0. \tag{25}$$

In order to obtain λ_F^c we need to have an analytical expression for F that in turn depends on both A and D (see (19)). When we were in the non-cooperative setting it was possible to characterize analytically the solution to forest owners' problem by supposing *ex-ante* that we were in the right case of figure, and then verifying, *ex-post*, that our first order conditions were indeed satisfied (see Appendix B for more details). This type of reasoning was possible because the optimal afforestation and deforestation rates were constant. In the present case, however, we can have a policy switch on A and/or D at any time. The value of λ_F^c is a function of the switching time on A and D. Thus, the first-order conditions will be satisfied for all $t \in [0, T]$ only if the exact switching time for both variables is chosen.

With respect to the speed of adjustment of emissions, V, the Lagrangian function is linear and $\frac{\partial \mathcal{L}^c}{\partial V} = \lambda_E^c E$. Thus, the optimal speed of adjustment of emissions is a bang-bang policy as follows:

$$V(t) = \begin{cases} V_{\min} & \text{if} \quad \lambda_E^c(t)E(t) < 0, \\ \widetilde{V} \in [V_{\min}, V_{\max}] & \text{if} \quad \lambda_E^c(t)E(t) = 0, \\ V_{\max} & \text{if} \quad \lambda_E^c(t)E(t) > 0. \end{cases}$$

We need to know λ_E^c to derive the optimal cooperative emissions strategy. Equations (21) and (22) can be written as:

$$\dot{\lambda}_S^c = r\lambda_S^c + 2c(S - \underline{S}), \ \ \lambda_S^c(T) = 2c\phi[\underline{S} - S(T)], \tag{26}$$

$$\dot{\lambda}_E^c = (r - V)\lambda_E^c - a + bE - \lambda_S^c, \quad \lambda_E^c(T) = 0. \tag{27}$$

From (27) we see that λ_E^c is a function of λ_S^c. And from (26) we have that S is a function of F. Therefore, to obtain λ_E^c we need to know the evolution of the forest stock and the evolution of the forest stock depends on the afforestation and deforestation policies applied. As it turns out, not only do we have a potential switch of regime for all our three controls, but the switches themselves are interdependent.

One can obtain the analytical expressions for the evolution of the state and co-state variables for all the possible cases of figure (i.e., before and after the switch). But just as it happened with non-forest owners' problem, it is not possible to derive the exact switching times analytically.

Denote now by t_A^c, t_D^c and t_V^c the switching time for our three control variables A, D and V respectively. We have evaluated the discounted intertemporal sum of joint payoffs for all the possible combinations of integer switching times (t_A^c, t_D^c, t_V^c) using a similar algorithm as before. See Table 2 for a sketch of the algorithm.

The only difference with respect to our previous algorithm is that now the computational complexity is increased as a consequence of the multiplicity of cases. Denote now by $\tilde{t}_A^c$, $\tilde{t}_D^c$, $\tilde{t}_V^c$ the three integer switching times that yield greater intertemporal payoffs. We have computed $\tilde{t}_A^c$, $\tilde{t}_D^c$, $\tilde{t}_V^c$ for $T = \{\mathbb{N} \in [1, 100]\}$ and for $r \in \{0.01,\ 0.02,\ 0.03\}$. Again, the results are linked to the length of the planning horizon and the discount rate used.

Table 2: Sketch of the algorithm used to obtain $\tilde{t}^c_\rho$, $\tilde{t}^c_D$, $\tilde{t}^c_v$

Fix the joint intertemporal discount rate r
for all integer planning horizons $T \in [1, ..., 100]$
  for all possible integer switching times t^c_ρ
    for all possible integer switching times t^c_D
      for all possible integer switching times t^c_v **do**

$$\begin{aligned}\text{JointPayoff}(t^c_\rho, t^c_D, t^c_v, T) &= \text{Discounted sum of revenues of FO}\\ &+ \text{Discounted sum of payoffs of NFO}\\ &+ \text{Scrap value function of NFO}\end{aligned}$$

      end
    end
  end
end
Select the t^c_ρ, t^c_D, t^c_v whose JointPayoff is greater for each value of T

Our results, call for the following comments: The solutions obtained can be classed into four different groups that coincide with four regions of the parameter space. We denote them by Z_1 to Z_4. The boundaries of regions $Z_1 - Z_4$ are related to parameter T. We denote the limits to these regions by T_1, T_1 and T_3. Figure 5 is a schematic representation of the solution.

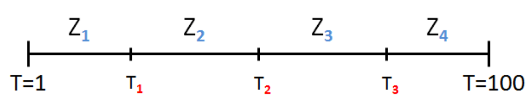

Figure 5: Cooperation timeline is a function of T

The results, that are summarized in Table 3, call for the following comments: (i) If the planning horizon is short (i.e., $T < T_1$) we are in region Z_1 and the cooperative solution coincides with the non-cooperative one (i.e., the cooperative solution brings no gain). The label not applicable ($N.A.$) is used here to denote that there is no switching time and that the solution coincides with the *status quo*. (ii) If we are in region Z_2 (i.e., $T_1 \leq T < T_2$) then it is jointly optimal to afforest at maximum rate for some time and then switch to afforestation $A_{\min}$ some time before the end of the planning horizon. It is not optimal to afforest all the time and we have that $A^* = A_{\max}$ if $t < \tilde{t}^c_A$ and $A^* = A_{\min}$ if $t \geq \tilde{t}^c_A$. We use the notation $\tilde{t}^c_A = f(T)$ to denote the fact that the switching time depends on T. In fact $f(T)$ is an increasing function of T. Clearly, for larger values of T it is optimal to switch later. The same reasoning applies for $\tilde{t}^c_D$. In this case, though, we have that $D^* = D_{\min}$ if $t < \tilde{t}^c_D$ and $D^* = D_{\max}$ if $t \geq \tilde{t}^c_D$. (iii) If we are in region Z_3 (i.e., $T_2 \leq T < T_3$) then it is optimal to apply $A^* = A_{\max}$ and $D^* = D_{\min}$ all along. We have used the notation $\tilde{t}^c_A = \tilde{t}_D = T$

to differentiate it from label $N.A.$ Recall that label $N.A.$ was used to denote that there is no switch and the optimal policy is identical to the *status quo* one (i.e., $A^* = A_{\min}$ and $D^* = D_{\max}\ \forall t \in [0, T]$) whereas in region Z_3 we have that there is no switch either, but the optimal policy is to apply $A^* = A_{\max}$ and $D^* = D_{\min}$ throughout. (iv) Finally, region Z_4 is identical to region Z_3 except for the emissions policy. If $T \geq T_3$ then it is certain that we will have a jump from $V_{\max}$ to $V_{\min}$ at some point in time $\tilde{t}_V^c$. The time of the switch is also a function of T.

Table 3: Jointly optimal policies are a function of T

Switch	Z_1	Z_2	Z_3	Z_4
$\tilde{t}_A$	$N.A.$	$\tilde{t}_A = f(T)$	T	T
$\tilde{t}_D$	$N.A.$	$\tilde{t}_D = g(T)$	T	T
$\tilde{t}_V$	$N.A.$	$N.A.$	$N.A.$	$\tilde{t}_V = h(T)$

Cooperation is more intense and the solution is more environmentally friendly as we move from region Z_1 (no cooperation) to region Z_4 (full cooperation and emissions abatement). When the discount rate is smaller, the environmental damage is further internalized. Table 4 shows the values of T_1 to T_3 for our different values in the discount rates. It is not surprising that when the discount rates are smaller the threshold planning horizons (T_1, T_2, T_3) between regions Z_1, Z_2, Z_3 and Z_4 are shifted downwards (See Table 4).

Table 4: Threshold times are a function of the discount rate

Discount	T_1	T_2	T_3
$r = 1\%$	11	19	36
$r = 2\%$	12	20	38
$r = 3\%$	12	21	41

4.1 Cooperation brings asymmetric results

We have showed that joint payoffs are greater in the cooperative setting provided that $T \geq T_1$. This is due to the damage reduction generated by increased afforestation effort and lower deforestation rates. Cooperation, however, does not bring gains to both players. Non-forest owners gain from the lower environmental damage, while forest owners lose by applying forest policies that are environmentally friendly but revenue harming.

Denote by $\pi_{NF}^c(x^c)$ the discounted sum of payoffs that non-forest owners get in the cooperative setting, where x^c denotes the state of the system along the cooperative trajectory. Analogously $\pi_{NF}^{nc}(x^{nc})$ denotes the discounted sum of payoffs that non-forest owners get in the non-cooperative setting.

The difference $\pi_{NF}^c(x^c) - \pi_{NF}^{nc}(x^{nc})$ measures the individual gain that non-forest owners obtain from cooperation. By the same token $\pi_{FO}^{nc}(x^{nc}) - \pi_{FO}^c(x^c)$

represents the loss that forest owners have in the cooperative setting *vis-à-vis* the non-cooperative one. These two quantities are a function of T and r. We compare them in Figure 6 for $r = 2\%$.

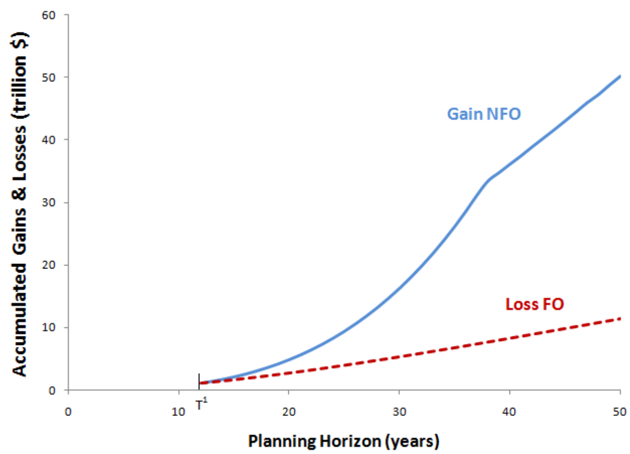

Figure 6: Cooperation gains and losses by NFO and FO

The cooperative gain by non-forest owners is represented by the solid line; while the loss by forest owners is represented by the dashed one. The vertical difference between these two lines measures the net cooperative gain for any given planning horizon. We observe that, for $T \geq T_1$ (with $T_1 = 11$ years)

$$\pi^c_{NF}(x^c) - \pi^{nc}_{NF}(x^{nc}) > \pi^{nc}_{FO}(x^{nc}) - \pi^c_{FO}(x^c).$$

It is not the issue of this paper to determine how this cooperative solution can be implemented. However, from Figure 6 it is clear that the implementation of the cooperative solution will require some sort of compensation from non-forest owners to forest-owners.

To sum up, the jointly optimal solution is different from the *status quo* one for T sufficiently long (i.e., $T \geq T_1$) and involves *greener* outcomes. If the planning horizon is short but not too long (i.e., $T_1 \leq T < T_3$) it will be optimal to mitigate the damage by increasing afforestation and decreasing deforestation, but not to abate emissions. If the planning horizon is sufficiently long (i.e. $T \geq T_3$) then it will be optimal both to mitigate (from the beginning) and to abate emissions (from time $\tilde{t}^c_V$ onwards). Emissions abatement has a greater cost than increasing afforestation or decreasing deforestation, this explains why it is preferable to start by applying less costly measures first and then move into more drastical changes as environmental damages increase.

4.2 Robustness analysis

Most of the parameters used in the forestry model proposed for forest owners have been obtained from FAO's Forest Resources Assessment (2006) or, when unavailable at FAO, from other sources in the literature (see Appendix A). Parameters a and b used in non-forest owners' payoff function have been calibrated to fit world GDP while making sure that emissions' gains are always increasing and concave. Finally, parameter c captures the environmental damage coming from the accumulation of greenhouse gases in the atmosphere and is key to the model. There is great uncertainty as to what is the exact impact of emissions on climate change and, therefore, on damage. For this reason attempting to estimate parameter c is a hard task. In an effort to account for part of this uncertainty on our results we have performed a sensitivity analysis with respect to it and analyzed two cases of figure. First, in Case 1 we suppose the environmental damage parameter (c) to be one third greater than the benchmark case used so far. Then, in Case 2 we suppose c to be one third below the same benchmark.

We have recomputed the cooperative outcomes obtained in the previous section for these two cases of figure. We do not observe qualitative changes. Just as before, we have four areas of interest $Z_1 - Z_4$. The behaviour in these four areas is exactly the same. The only difference that we observe is that cooperation will be more easily (i.e. earlier) achieved when the environmental damage is higher (Case 1). The results are summarized in Table 5 below:

Table 5: Robustness of solution to changes in environmental damages

Discount	Benchmark			Case 1			Case 2		
	T_1	T_2	T_3	T_1	T_2	T_3	T_1	T_2	T_3
$r = 1\%$	11	19	36	8	15	31	15	25	42
$r = 2\%$	12	20	38	9	16	33	17	27	46
$r = 3\%$	12	21	41	9	17	36	18	29	49

Table 5 shows the threshold times ($T_1 - T_3$) for Case 1 and 2. These thresholds are shifted downwards when c increases (Case 1) and upwards when c decreases (Case 2). A downward shift in T_1 indicates that the minimum planning horizon beyond which cooperation brings gains is reduced. Analogously, if T_2 and T_3 are reduced this will mean that it is optimal to enhance cooperation for shorter planning horizons than before, and vice versa for upwards shifts of the thresholds.

To sum up, our results seem quite robust to changes in parameter c and, even if the thresholds are affected, the structure of the solutions does not change and cooperation is still strictly welfare improving for all the scenarios studied regardless of the value of c used.

5 Conclusions

Forests play an important role in mitigating climate change. In this paper we have proposed a two-player model where forest owners have an incentive to deforest so as to increase their economic revenues; while non-forest owners suffer a negative externality coming from deforestation due to the so called *reduced-carbon-sequestration effect* that states that a tree that is cut cannot grow and hence cannot sequester carbon. We model the economic incentives of both types of players and explore the conditions that make environmental cooperation strictly welfare improving. We show that longer time horizons and smaller discount rates help to better account for greenhouse gas accumulation damage.

We have proposed three different mechanisms to reduce GHG accumulation: abatement of emissions, increases in afforestation, and decreases in deforestation.

We show that for short planning horizons cooperation brings no gain. For longer planning horizons it is jointly optimal to have some afforestation effort and deforestation reduction. Cooperation brings tangible economic gains that increase with the length of the planning horizon. For even longer planning horizons it is optimal to combine forestation efforts with emissions abatement. Reducing emissions is more expensive but also more effective in offsetting the environmental damage coming from the excessive accumulation of GHGs.

Cooperation along with a sufficiently long planning horizon allows to partly internalize the positive externality that the carbon sequestration by forests creates. Cooperation brings *greener* outcomes because it helps mitigate climate change and slows down forest depletion.

Our results also convey a double positive message: First, considering the carbon sequestration potential of forests can make a significant difference to stop forest destruction. Second, international cooperation can bring sound economic and environmental gains.

The results obtained in this paper are very promising. However, there are many aspects that have not been considered and call for a critical interpretation of the results: (i) A more comprehensive dynamics of the accumulation of greenhouse gases should consider emissions related to land use change. (ii) Carbon sequestration by the oceans may be affected by the excessive acidification of the oceans. A more thorough research should integrate this aspect. (iii) We have analyzed when is it that cooperation strictly improves joint welfare. However, nothing has been said on how this cooperative solution could be implemented nor on how the surplus arising from it would be distributed. (iv) One could envisage some sort of transfer mechanism as a way to ensure cooperation. In that case it would be interesting to study the time consistency of the cooperative solution with transfers.

Appendix A: Variables & parameters description

State variables

F: **Stock of Forest**

Forest surface area of the world measured in hectares. The current stock of forest F_0 is estimated by FAO (2006) at 3952 million hectares in 2005. Parameter $\bar{F} = F_{\max}$ has been estimated for 1750 AD at 42% of the globe's surface[10] (i.e., 13067 million hectares excluding Antarctica and Greenland). This gives us a value for $F_{\max}$ of approximately 5500 million hectares. Consequently, we require that $F(t) \in [F_{\min}, F_{\max}] = \left[0 \cdot 10^9,\ 5.5 \cdot 10^9\right]$.

E: **Yearly emissions of CO_2**

These are world yearly emissions of CO_2 measured in metric tons. In 2005, total CO_2 emissions (E_0) amounted to 28.2 billion metric tons, i.e., 7.7 GtC (gigatons of carbon).[11]

S: **Cumulated quantity of CO_2 in the atmosphere**

The cumulated stock of CO_2 in the atmosphere is measured in gigatons. The current stock of CO_2 that we denote by S_0 has been estimated to be approximately at 3000 Gt of CO_2 that are equivalent to 800 GtC (383-387 ppmv) in 2007.[12]

Control variables

A : **Yearly afforestation** $[A_{\min}, A_{\max}] = [0,\ 3 \cdot 10^6]$

For the period 1990-2005, FAO (2006) estimates world yearly afforestation at 2.8 million hectares.

D : **Yearly deforestation** $[D_{\min}, D_{\max}] = [0,\ 13 \cdot 10^6]$

For the same period, FAO (2006) estimates the average global deforestation rate at 13 million hectares per year.

V : **CO_2 emissions adjustment rate** $[V_{\min}, V_{\max}] = [-0.015,\ 0.03]$

During the decade going from 1990 to 2000 world CO_2 emissions increased roughly 1% every year.[13] Only a few countries (e.g., Germany, United Kingdom, Denmark, Finland, some Eastern European countries and former Soviet Republics) were able to reduce their emissions. Germany was the most outstanding case and achieved a 1.8% yearly cumulative decrease. On the other hand China's emissions increased at a rate of 3% per annum. All the other big economies lie somewhere in between. Between 2000 and 2008, however, world emissions have

[10] Source: http://www.geo.vu.nl/~renh/deforest.htm

[11] Source: EIA (2008).

[12] Source: NOAA (2007).

[13] Source: Bernstein et al. (2006) and EIA (2008).

increased at a faster rate, 3.4% yr^{-1}, (Le Quéré et al., 2009) with a probable decrease during the next two years (2009-2010) due to the world crisis. Following these observations we have set both the lower and upper bounds on V. As a benchmark scenario we have chosen $V \in [V_{\min}, V_{\max}] = [-0.015, 0.03]$.

Parameters

a, b: Emissions-output ratio parameters

Parameters a and b are chosen to ensure that (i) $G(E)$ in 8 is increasing and concave throughout and (ii) $G(E_0)$ equals world's GDP estimate by the World Bank for the year 2008.[14] $a = 2100$, $b = 4 \cdot 10^{-9}$.

c: Environmental damage parameter

This parameter captures the impact of greater GHG concentration levels on the welfare of individuals. $c = 1.5 \cdot 10^{-11}$.

$\underline{S}$: Pre-industrial CO_2 concentration level

Parameter $\underline{S}$ has been set to match preindustrial levels, i.e., 284 ppmv in year 1832[15] that are equivalent to 587 GtC.

κ_1: Per hectare afforestation cost

The World Bank estimates the cost for seedling at roughly 40 US$ per thousand seedlings. The number of seedlings per hectare is equal to approximately 2000. This amounts to approximately 80 $ per hectare of forest in terms of seedling. Afforestation costs also include other costs (e.g., labour costs) that fluctuate with countries. The World Agroforestry Centre gives estimates for the Philippines around 1000 $/ha. Other NGO organisations provide estimates that range between 180 $/ha for Senegal and 400-500 $/ha for other countries in Africa such as Sudan, Madagascar and Ethiopia.[16] We have chosen the round and representative value of 500 $/ha.

κ_2: Per hectare deforestation cost

The Bureau of Business and Economic Research of Montana University estimates the costs of ground-based logging per green ton of harvest for the year 2006 at 22.70$.[17] A green ton is equivalent 907 kg (2000 pounds of undried biomass material). The density of wood is typically 500 kg/m^3. For a representative douglas fir plantation (530 kg/m^3) we obtain a deforestation cost of 13.26 $\$/m^3$. If the yield per hectare, is equal to 110 m^3/ha (see the estimation of n below) then we obtain an estimate of the deforestation cost per hectare of 1459 $/ha.

[14] http://web.worldbank.org/
[15] Source: NOAA.
[16] See e.g., www.villageprojectsint.org and www.edenprojects.org
[17] www.bber.umt.edu/pubs/forest/prices/loggingCostPoster.pdf

η: **Natural growth rate of the forest**

FAO (2006) estimates the average yearly natural expansion of world forests to be equal to 2.9 million hectares, i.e., $\eta\ F = 2.9 \cdot 10^6$ ha. Parameter η is dimensionless and can be estimated accordingly: $\eta = 7.34 \cdot 10^{-4}$.

φ: **Carbon absorption rate**

This parameter is measured in metric tons of CO_2 equivalent absorbed per hectare of forest and year. According to Le Quéré et al. (2009) during the decade from 1990-2000 forests absorbed 2.6 PgC yr^{-1}(i.e., 2.6 GtC) which amounts to 9.53 $GtCO_2$. World total forest area equals 3952 million hectares. If we consider a homogeneous forest, its mean yearly carbon sequestration is 2.412 tonnes of CO_2 $ha^{-1}yr^{-1}$.

W: **Carbon absorption rate by oceans**

Le Quéré et al. (2009) estimate that oceans were able to sink, on average, 2.2 PgC yr^{-1}(8.07 $GtCO_2$ yr^{-1}) during the period 1990-2000. We have set parameter W equal to their estimate.

n: **Per hectare timber yield**

Timber yield is measured in m^3 of wood per hectare. According to FAO (2006) the mean wood content of a hectare of forest land in 2005 is equal to 110 m^3.

β: **Lower productivity due to forest depletion**

Eswaran et al. (2001) estimate the productivity loss as a consequence of land degradation, erosion, and desertification for the African continent at 8.2% of average productivity. Average land productivity is measured by $\overline{x}$ (see the estimation below). Parameter β is thus equal to 0.061 (8.2% of $\overline{x}$).

γ: **Selective logging yield, fraction of average yield**

The selective logging yield is measured as a fraction of average yield. When forests are managed for wood production they produce as much as 1-3 m^3 per hectare (in other words $n\gamma$ =1-3 m^3). Following Andrés-Domenech et al (2011) we set the value of $\gamma = 1.5\%$.

δ: **Fraction of forests selectively logged**

Share of the world's forests selectively-logged. Following FAO (2006), parameter δ has been calibrated at 30% to fit the current world yearly production of wood.

θ: **Slope of wood demand**

According to FAO (2006), the commercial value of all wood (i.e., roundwood and fuelwood) in 2005 was US$64 billion per year of which only 7 billion correspond to fuelwood. Current world production equals 3400 million m^3. The average price for both types of wood is 18.8 dollars per m^3. FAO (1997) gives the elasticity of demand for several countries and several types of wood.[18] A representative value of both the mean and median price elasticity of wood is -0.50. We have approximated an iso-elastic curve by a linear one in an interval of 2000 million m^3 centred at 3400 million m^3 such that the average elasticity inside the interval equals -0.50. The slope of our demand can be then computed and we obtain $\theta = -2.7 \cdot 10^{-9}$.

$\overline{p}$: **Choke price of wood**

With the average price of wood and the slope of demand computed above, we can retrieve the choke price of our inverse demand function and obtain $\overline{p} = 27.98$ (US$ per m^3).

ψ: **Extra productivity of deforested land**

Parameter ψ denotes the productivity gain of land after deforestation. It is measured as a fraction of average productivity. We adopt $\psi = 0.3$ following Andrés-Domenech et al. (2011).

p_A: **Average price of representative agricultural product**

Measured in US$ per metric ton. To determine the average price of the representative agricultural good we took four representative commodities (i.e., cocoa, coffee, cotton and sugar) from FAO (2004). These four commodities are related to deforestation processes. The net economic yield per hectare of crop ranges from 1660 $/ha for coffee to 771 $/ha for cocoa. The mean yield equals 1141 $/ha. Cotton is the more representative of the four in terms of prices and economic yield (1467 $ per metric ton and 1088 $/ha). We use the price of cotton as a reference.

$\overline{x}$: **Average land productivity**

Measured in tons per hectare. Average land productivity has been computed with the same four crops used to obtain $p_A \cdot \overline{x}$ is set equal to 0.742 metric tons per hectare.

Appendix B: Proof of proposition 1

The Hamiltonian of forest owners' control problem is:[19]

$$\begin{aligned} H^{FO}(F, A, D, \lambda) &= [\overline{p} - \theta(nD + n\gamma\delta F)]\, n(D + \gamma\delta F) \\ &+ p_A \left[\overline{x} + \frac{\psi\overline{x}}{\bar{F} - F} D - \beta \frac{\bar{F} - F}{\bar{F}}\right] \left(\bar{F} - F\right) - \kappa_1 A - \kappa_2 D \\ &+ \lambda\left[A + \eta F - D\right], \end{aligned}$$

[18] Most elasticity values are comprehended between -0.25 and -0.75

[19] The time argument is eliminated when no confusion can arise.

where λ denotes the co-state variable associated with the forest stock. The Lagrangian of forest owners can be written as:

$$\mathcal{L}^{FO}\left(F, A, D, \lambda, w_1, w_2\right) = H^{FO}\left(F, A, D, \lambda\right) + w_1 D + w_2\left(D_{\max} - D\right),$$

where $w_1(t)$ and $w_2(t)$ are the Lagrange multipliers associated with the non-negativity condition $D(t) \geq 0$ and $D(t) \leq D_{\max}$.[20]

The first-order optimality conditions read:

$$\max_{\substack{0\leq A\leq A_{\max}\\ 0\leq D\leq D_{\max}}} \mathcal{L}^{FO}\left(F, A, D, \lambda, w_1, w_2\right), \tag{28}$$

$$\begin{aligned}
\dot{F} &= A + \eta F - D, \quad \bar{F} \geq F \geq 0,\ F(0) = F_0, && (29)\\
\dot{\lambda} &= r_{FO}\lambda - \frac{\partial \mathcal{L}^{FO}}{\partial F}, \quad \lambda(T) = 0, && (30)\\
w_1 &\geq 0,\ w_1 D = 0,\ D \geq 0,\\
w_2 &\geq 0,\ w_2(D_{\max} - D) = 0,\ D_{\max} \geq D.
\end{aligned}$$

The necessary condition for the maximization problem in (28) with respect to the deforestation rate reads:

$$\frac{\partial \mathcal{L}^{FO}}{\partial D} = 0; \quad -2\theta n^2\left[\gamma\delta F + D\right] + \overline{p}n + p_A\psi\overline{x} + w_1 - w_2 - \kappa_2 - \lambda = 0. \tag{31}$$

With respect to the afforestation rate, A, we have a Lagrangian that is linear in A and $\frac{\partial \mathcal{L}^{FO}}{\partial A} = -\kappa_1 + \lambda$. The optimal afforestation rate is a bang-bang policy as follows:

$$A^*(t) = \begin{cases} 0 & \text{if} \quad -\kappa_1 + \lambda(t) < 0,\\ \widetilde{A} \in [0, A_{\max}] & \text{if} \quad -\kappa_1 + \lambda(t) = 0,\\ A_{\max} & \text{if} \quad -\kappa_1 + \lambda(t) > 0. \end{cases} \tag{32}$$

The differential equation (30) for the co-state variable reads:

$$\begin{aligned}
\dot{\lambda} &= (r_{FO} - \eta)\lambda + 2\theta n^2\gamma\delta(D + \gamma\delta F) + 2p_A\beta\frac{F}{\overline{F}} + p_A(\overline{x} - 2\beta) - \overline{p}n\gamma\delta,\\
\lambda(T) &= 0. && (33)
\end{aligned}$$

For the values of our parameters it can be proved that a maximum deforestation rate ($D(t) = D_{\max}$ for all $t \in [0, T]$) and a minimum afforestation rate ($A(t) = 0$ for all $t \in [0, T]$) satisfy the optimality conditions established above. Replacing these optimal policies into the dynamics of the forest stock given in (29) we have:

$$\dot{F} = \eta F - D_{\max}, \quad F(0) = F_0.$$

[20]To simplify the notation, we do not include Lagrange multipliers associated with the non-negativity conditions on the other control variable, $\rho(t)$, because this variable enters in a linear way in the model and the optimal afforestation policy is bang-bang.

The solution to this differential equation is given by (14). Plugging (14) in equation (33) leads to:

$$\begin{aligned}\dot{\lambda} &= (r_{FO} - \eta)\lambda - \overline{p}n\gamma\delta + 2\theta n^2\gamma\delta D_{\max} + p_A(\overline{x} - 2\beta) \\ &+ 2\left[\left(F_0 - \frac{D_{\max}}{\eta}\right)e^{\eta t} + \frac{D_{\max}}{\eta}\right]\left[\theta n^2\gamma^2\delta^2 + p_A\beta\frac{1}{\overline{F}}\right], \quad \lambda(T) = 0.\end{aligned}$$

From the integration of the above non-homogeneous linear differential equation we get the following

$$\begin{aligned}\lambda(t) &= \frac{1}{\eta - r_{FO}}\left\{-\overline{p}n\gamma\delta + 2\theta n^2\gamma\delta D_{\max} + p_A(\overline{x} - 2\beta)\right. \\ &+ \left. 2\left[\left(F_0 - \frac{D_{\max}}{\eta}\right)e^{\eta t} + \frac{D_{\max}}{\eta}\right]\left[\theta n^2\gamma^2\delta^2 + p_A\beta\frac{1}{\overline{F}}\right]\right\} + K_\lambda e^{(r_{FO} - \eta)t},\end{aligned}$$

where K_λ denotes the constant of integration.

This constant K_λ can be retrieved using the transversality condition for the co-state variable λ, $\lambda(T) = 0$. The final expression of the co-state optimal time path reads as in (15).

For our parameter domain λ always takes negative values and increases over time to reach zero at T. Therefore, from (32), we conclude that the optimal afforestation policy is $A(t) = 0$ for all $t \in [0, T]$.

$$\frac{\partial \mathcal{L}^{FO}}{\partial D} = 0; \quad -2\theta n^2\left[\gamma\delta F + D\right] + \overline{p}n + p_A\psi\overline{x} + w_1 - w_2 - \kappa_2 - \lambda = 0.$$

Finally, to show that the optimal deforestation rate D^* is indeed $D_{\max}$ for all $t \in [0, T]$ (and hence $w_1 = 0$ and $w_2 \neq 0$), we replace the optimal time paths of F and λ given by (14) and (15), respectively, in equation (31). Given our parameters' values we observe that if $w_2 = 0$, then the LHS of equation (31) is positive -instead of null- for all feasible F and D. The only way to avoid this contradiction is by having $w_2 \neq 0$. In other words, forest owners maximize their payoffs for $D = D_{\max}$ and the forest area along the optimal path decreases with time.

Appendix C: Proof of proposition 2

The Hamiltonian of the optimal control problem of non-forest owners is:

$$\begin{aligned}H^{NF}(F, S, E, V, \lambda_F, \lambda_S, \lambda_E) &= aE - \frac{1}{2}bE^2 - c\left(S - \underline{S}\right)^2 \\ &+ \lambda_F\left[A + \eta F - D\right] + \lambda_E VE + \lambda_S\left[E - \varphi F - W\right].\end{aligned}$$

The first-order optimality conditions read:[21]

$$
\begin{aligned}
&\max_V H^{NF}, \\
\dot{F} &= A + \eta F - D, \quad \bar{F} \geq F \geq 0,\ F(0) = F_0, \\
\dot{S} &= E - \varphi F - W, \quad S \geq 0,\ S(0) = S_0, \\
\dot{E} &= VE, \quad E \geq 0,\ E(0) = E_0,\ V \in [V_{\min}, V_{\max}] && (34) \\
\dot{\lambda}_F &= r_{NF}\lambda_F - \frac{\partial H^{NF}}{\partial F}, \quad \lambda_F(T) = 0, && (35) \\
\dot{\lambda}_S &= r_{NF}\lambda_S - \frac{\partial H^{NF}}{\partial S}, \quad \lambda_S(T) = -\frac{d\phi\,(S(T))}{dS(T)}, && (36) \\
\dot{\lambda}_E &= r_{NF}\lambda_E - \frac{\partial H^{NF}}{\partial E}, \quad \lambda_E(T) = 0. && (37)
\end{aligned}
$$

Since the Hamiltonian is linear in V, condition (34) and $\frac{\partial H^{NF}}{\partial V} = \lambda_E E$, lead to the following optimal bang-bang solution:

$$
V^*(t) = \begin{cases} V_{\min} & \text{if} \quad \lambda_E(t)E(t) < 0, \\ \widetilde{V} \in [V_{\min}, V_{\max}] & \text{if} \quad \lambda_E(t)E(t) = 0, \\ V_{\max} & \text{if} \quad \lambda_E(t)E(t) > 0. \end{cases}
$$

Equations (35), (36) and (37) can be written as:

$$
\begin{aligned}
&\dot{\lambda}_F = (r_{NF} - \eta)\lambda_F + \varphi\lambda_S, \quad \lambda_F(T) = 0, \\
&\dot{\lambda}_S = r_{NF}\lambda_S + 2c(S - \underline{S}), \quad \lambda_S(T) = 2c\phi[\underline{S} - S\,(T)], && (38) \\
&\dot{\lambda}_E = (r_{NF} - V)\lambda_E - a + bE - \lambda_S, \quad \lambda_E(T) = 0, && (39)
\end{aligned}
$$

where

$$
\phi = \frac{1}{r_{NF}}\left(1 - e^{-r_{NF}T}\right).
$$

Let us assume $V(t) = V^*$ constant over the planning horizon, where V^* denotes either $V_{\min}, V_{\max}$ or $\widetilde{V}$. Solving the differential equation in (34) we can characterize the optimal trajectory of emissions, $E(t)$, which is given by (16).

From the problem of forest owners, the optimal path of the forest stock is known and given by equation (14). Take equations (14) and (16), and plug them in (34). Integration of the resulting expression gives the expression in (17).

Given our parameter domain, it can be shown that both the optimal paths of emissions and stock of greenhouse gases are always greater than zero.

Using the expressions for the optimal paths of the state variables $F(t)$, $E(t)$ and $S(t)$ (expressions (14), (16) and (17) respectively), we can retrieve the optimal paths of the three co-state variables.

[21] In order to simplify the presentation we do not explicitly introduce the Lagrangian function and the restrictions on the state variables, but we check a posteriori that all these restrictions are satisfied. The time argument is also eliminated when no confusion can arise.

From the integration of the differential equation of the shadow price of the pollution stock, λ_S, in (38), we get:

$$\lambda_S(t)= K_S e^{r_{NF} t} - \frac{2c}{r_{NF}} \left[S_0 - \underline{S} - \frac{E_0}{V^*} + \frac{\varphi}{\eta} \left(F_0 - \frac{D_{\max}}{\eta} \right) \right]$$
$$+2c\left[\frac{1}{r_{NF}}\left(t+\frac{1}{r_{NF}}\right)\left(W+\frac{\varphi}{\eta}D_{\max}\right)-\frac{\varphi}{\eta}\left(F_0-\frac{D_{\max}}{\eta}\right)\frac{e^{\eta t}}{\eta-r_{NF}}+\frac{E_0}{V^*}\frac{e^{V^* t}}{V^*-r_{NF}}\right],$$

where K_S denotes the constant of integration. This constant can be easily determined using the transversality condition $\lambda_S(T) = 2c\phi[\underline{S} - S(T)]$. After replacing this constant on λ_S the optimal path of the shadow price of the pollution stock reads:

$$\lambda_S(t) = \Lambda_1 + \Lambda_2 e^{-r_{NF}(T-t)} + \Lambda_3 t + \Lambda_4 e^{V^* t} + \Lambda_5 e^{\eta t},$$

where:

$$\begin{aligned}
\Lambda_1 &= -\frac{2c}{r_{NF}} \left[S_0 - \underline{S} - \frac{E_0}{V^*} + \frac{\varphi}{\eta} \left(F_0 - \frac{D_{\max}}{\eta} \right) - \frac{1}{r_{NF}} \left(W + \frac{\varphi}{\eta} D_{\max} \right) \right], \\
\Lambda_2 &= -\Lambda_1 - 2c\left[\left(\frac{W}{r_{NF}}+\frac{\varphi}{\eta}\frac{D_{\max}}{r_{NF}}\right)T-\frac{\varphi}{\eta}\left(F_0-\frac{D_{\max}}{\eta}\right)\frac{1}{\eta-r_{NF}}e^{\eta T}+\frac{E_0}{V^*}\frac{e^{V^*T}}{V^*-r_{NF}}\right] \\
&\quad +2c\phi[\underline{S} - S(T)], \\
\Lambda_3 &= \frac{2c}{r_{NF}} \left(W + \frac{\varphi}{\eta} D_{\max} \right), \\
\Lambda_4 &= 2c \frac{E_0}{V^*(V^* - r_{NF})}, \\
\Lambda_5 &= -2c\frac{\varphi}{\eta} \left(F_0 - \frac{D_{\max}}{\eta} \right) \frac{1}{\eta - r_{NF}}, \\
S(T) &= S_0 - WT - \frac{\varphi}{\eta} \left(D_{\max} T - \left(F_0 - \frac{D_{\max}}{\eta} \right) (1 - e^{\eta T}) \right) - \frac{E_0}{V^*} (1 - e^{V^* T}).
\end{aligned}$$

Once we have λ_S we can plug it in expression (39) to obtain λ_E. Integrating the resulting expression gives:

$$\lambda_E(t) = \frac{a}{r_{NF} - V^*} - \frac{bE_0}{r_{NF} - 2V^*} e^{V^* t} + \frac{\Lambda_1}{r_{NF} - V^*} + \frac{\Lambda_2}{V^*} e^{-r_{NF}(T-t)} e^{-2V^* t}$$
$$+\frac{\Lambda_3}{r_{NF}-V^*}\left(t+\frac{1}{r_{NF}-V^*}\right)+\frac{\Lambda_4}{r_{NF}-2V^*}e^{V^* t}+\frac{\Lambda_5}{r_{NF}-V^*-\eta}e^{\eta t}+K_E e^{(r_{NF}-V^*)t},$$

where K_E denotes the constant of integration. To determine K_E we use the transversality condition $\lambda_E(T) = 0$, and substitute its value in the expression above. The co-state variable $\lambda_E(t)$ then reads as in expression (40).

$$\begin{aligned}\lambda_E(t) = &(\Lambda_1+a)\frac{1}{r_{_{NF}}-V^*}(1-\Gamma(t)) - \frac{bE_0}{r_{_{NF}}-2V^*}\left(e^{V^*t}-e^{V^*T}\Gamma(t)\right)\\ &+ \frac{\Lambda_2}{V^*}[e^{-2V^*t}e^{-r_{_{NF}}(T-t)} - e^{-2V^*T}\Gamma(t)]\\ &+ \frac{\Lambda_3}{r_{_{NF}}-V^*}\left(t+\frac{1}{r_{_{NF}}-V^*} - \left(T+\frac{1}{r_{_{NF}}-V^*}\right)\Gamma(t)\right)\\ &+ \frac{\Lambda_4}{r_{_{NF}}-2V^*}(e^{V^*t}-e^{V^*T}\Gamma(t)) + \frac{\Lambda_5}{r_{_{NF}}-V^*-\eta}(e^{\eta t}-e^{\eta T}\Gamma(t)).\end{aligned} \tag{40}$$

where $\Gamma(t) = e^{-(r_n-V^*)(T-t)}$.

The optimal path for the shadow price of the forest stock $\lambda_F(t)$ can be obtained analogously and is given by expression (41).

$$\begin{aligned}\lambda_F(t) = \varphi\Bigg[&\left(-\frac{\Lambda_1}{r_n-\eta}+\frac{\Lambda_2}{\eta}\Psi(t)\right)(1-\Psi(t)) - \frac{\Lambda_4}{r_n-\eta-V^*}(e^{V^*t}-\Psi(t))\\ -&\frac{\Lambda_3}{r_n-\eta}\left(t+\frac{1}{r_n-\eta}-\left(T+\frac{1}{r_n-\eta}\right)\Psi(t)\right) - \frac{\Lambda_5}{r_n-2\eta}(e^{\eta t}-\Psi(t)e^{-\eta T})\Bigg],\end{aligned} \tag{41}$$

where $\Psi(t) = e^{-(r_n-\eta)(T-t)}$.

Appendix D: Switching time

If there is only one switch in the optimal policy (switch at time $\tilde{t}_V$) then the jump should be of the following type: First apply $V_{\max}$ $\forall t \in [0, \tilde{t}_V]$ and then apply $V_{\min}$ $\forall t \in [\tilde{t}_V, T]$. Applying $V_{\max}$ always brings greater yields in the short run than it does $V_{\min}$ and if one optimizes using a positive discount rate it is better to allocate emissions at the beginning of the planning horizon.

Recall that in absence of switching time we have that $S(t)$ is given by (17). Whereas, when there is a switch, the optimal expression for $S(t)$ changes. We now have a two-part expression, one before the switch and another afterwards.

$$S(t)=S_0-tW-\frac{\varphi}{\eta}D_{\max}t-\frac{E_0}{V_{\max}}(1-e^{V_{\max}t})+\frac{\varphi}{\eta}\left(F_0-\frac{D_{\max}}{\eta}\right)(1-e^{\eta t}), \forall t \in [0, \tilde{t}_V),$$

$$\begin{aligned}S(t)=&S(\tilde{t}_V) - \left(t-\tilde{t}_V\right)\left(W+\frac{\varphi}{\eta}D_{\max}\right) - \frac{E_0}{V_{\min}}e^{V_{\max}\tilde{t}_V}(1-e^{V_{\min}\left(t-\tilde{t}_V\right)})\\ &+\frac{\varphi}{\eta}\left(F(\tilde{t}_V)-\frac{D_{\max}}{\eta}\right)e^{\eta\tilde{t}_V}(1-e^{\eta\left(t-\tilde{t}_V\right)}), \quad \forall t \in [\tilde{t}_V, T],\end{aligned}$$

where

$$S(\tilde{t}_V)=S_0-\tilde{t}_VW-\frac{\varphi}{\eta}D_{\max}\tilde{t}_V-\frac{E_0}{V_{\max}}(1-e^{V_{\max}\tilde{t}_V})+\frac{\varphi}{\eta}\left(F_0-\frac{D_{\max}}{\eta}\right)(1-e^{\eta\tilde{t}_V}).$$

These expressions are straightforward to obtain considering that:

$$E(t) = \begin{cases} E_0 e^{V_{\max} t}, & \forall t \in [0, \tilde{t}_V] \\ E_0 e^{V_{\max} \tilde{t}_V} e^{V_{\min}(t-\tilde{t}_V)}, & \forall t \in [\tilde{t}_V, T]. \end{cases}$$

Once we have $S(t)$, $\lambda_S(t)$ can be computed using the transversality condition from the salvage value function. Proceeding similarly as we did to obtain $\lambda_S(t)$ in the case without switch, the following expression for the interval $[\tilde{t}_V, T]$ is obtained:

$$\lambda_S(t) = \Upsilon_1 + \Upsilon_2 e^{-r_{NF}(T-t)} + \Upsilon_3 t + \Upsilon_4 e^{V_{\min}\left(t-\tilde{t}_V\right)} + \Upsilon_5 e^{\eta t}, \tag{42}$$

where:

$$\begin{aligned}
\Upsilon_1 &= -\frac{2c}{r_{NF}}\left[S(\tilde{t}_V) - \underline{S} - \frac{E_0 e^{V_{\max}\tilde{t}_V}}{V_{\min}} + \frac{\varphi}{\eta}\left(F(\tilde{t}_V) - \frac{D_{\max}}{\eta}\right)\right] \\
&\quad -\frac{2c}{r_{NF}}\left(W + \frac{\varphi D_{\max}}{\eta}\right)\left(\tilde{t}_V - \frac{1}{r_{NF}}\right), \\
\Upsilon_2 &= -\Upsilon_1 - 2c\left[\frac{W}{r_{NF}}T + \frac{\varphi}{\eta}\left(\frac{D_{\max}}{r_{NF}}T - \left(F(\tilde{t}_V) - \frac{D_{\max}}{\eta}\right)\frac{1}{\eta - r_{NF}} e^{\eta(T-\tilde{t}_V)}\right)\right. \\
&\quad \left. + \frac{E_0}{V_{\min}} e^{V_{\max}\tilde{t}_V} \frac{e^{V_{\min}(T-\tilde{t}_V)}}{V_{\min} - r_{NF}}\right] + 2c\phi[\underline{S} - S(T)], \\
\Upsilon_3 &= \frac{2c}{r_{NF}}\left(W + \frac{\varphi}{\eta} D_{\max}\right), \\
\Upsilon_4 &= 2c\frac{E_0 e^{V_{\max}\tilde{t}_V}}{V_{\min}\left(V_{\min} - r_{NF}\right)}, \\
\Upsilon_5 &= -2c\frac{\varphi}{\eta}\left(F(\tilde{t}_V) - \frac{D_{\max}}{\eta}\right)\frac{1}{\eta - r_{NF}}.
\end{aligned}$$

Once $\lambda_S(t)$ $\forall t \in [\tilde{t}_V, T]$ is known, $\lambda_S(t)$ $\forall t \in [0, \tilde{t}_V]$ can be computed analogously and can be written in a compact manner as follows:

$$\lambda_S(t) = \Sigma_1 + \Sigma_2 e^{r_{NF}\left(t-\tilde{t}_V\right)} + \Sigma_3 t + \Sigma_4 e^{V_{\max} t} + \Sigma_5 e^{\eta t},$$

where

$$\Sigma_1 = -\frac{2c}{r_{NF}}\left[S_0 - \underline{S} - \frac{E_0}{V_{\max}} + \frac{\varphi}{\eta}\left(F_0 - \frac{D_{\max}}{\eta}\right) - \left(\frac{W}{r_{NF}} + \frac{\varphi}{\eta}\frac{D_{\max}}{r_{NF}}\right)\right],$$

$$\Sigma_2 = -\Sigma_1 - 2c\left[\left(\frac{W}{r_{NF}} + \frac{\varphi}{\eta}\frac{D_{\max}}{r_{NF}}\right)\tilde{t}_V - \frac{\varphi}{\eta}\left(F_0 - \frac{D_{\max}}{\eta}\right)\frac{e^{\eta\tilde{t}_V}}{\eta - r_{NF}} + \frac{E_0}{V_{\max}}\frac{e^{V_{\max}\tilde{t}_V}}{V_{\max} - r_{NF}}\right]$$
$$+\lambda_S(\tilde{t}_V),$$

$$\Sigma_3 = \frac{2c}{r_{NF}}\left(W + \frac{\varphi}{\eta}D_{\max}\right),$$

$$\Sigma_4 = 2c\frac{E_0}{V_{\max}\left(V_{\max} - r_{NF}\right)},$$

$$\Sigma_5 = -2c\frac{\varphi}{\eta}\left(F_0 - \frac{D_{\max}}{\eta}\right)\frac{1}{\eta - r_{NF}},$$

and where $\lambda_S(\tilde{t}_V)$ is the boundary condition to this problem that can be obtained by substituting for time $t = \tilde{t}_V$ in equation (42).

Now $\lambda_S(t)\ \forall t \in [\tilde{t}_V, T]$ and $\lambda_S(t)\ \forall t \in [0, \tilde{t}_V]$ are known; $\lambda_E(t)\ \forall t \in [\tilde{t}_V, T]$ and $\lambda_E(t)\ \forall t \in [0, \tilde{t}_V]$ can be obtained analogously. In this case it is easier since the boundary condition for λ_E (i.e., $\lambda_E(T)$) is equal to zero. The expression of $\lambda_E(t)\ \forall t \in [\tilde{t}_V, T]$ reads:

$$\lambda_E(t) = \frac{a}{r_{NF} - V_{\min}} - \frac{bE_0 e^{V_{\max}\tilde{t}_V}}{r_{NF} - 2V_{\min}} e^{V_{\min}\left(t - \tilde{t}_V\right)} + \frac{\Upsilon_1}{r_{NF} - V_{\min}} - \frac{\Upsilon_2}{V_{\min}} e^{-r_{NF}(T-t)}$$
$$+ \frac{\Upsilon_3}{r_{NF} - V_{\min}}\left(t + \frac{1}{r_{NF} - V_{\min}}\right) + \frac{\Upsilon_4}{r_{NF} - 2V_{\min}} e^{V_{\min}\left(t - \tilde{t}_V\right)}$$
$$+ \frac{\Upsilon_5}{r_{NF} - V_{\min} - \eta} e^{\eta\left(t - \tilde{t}_V\right)} + K_E e^{(r_{NF} - V_{\min})t}.$$

The constant of integration K_E can be obtained using the transversality condition $\lambda_E(T) = 0$. Denote $\Pi(t) = e^{(r_{NF} - V_{\min})(t-T)}$, then the value of $\lambda_E(t)\ \forall t \in [\tilde{t}_V, T]$ can be written as follows:

$$\begin{aligned}
\lambda_E(t) = &\ (a + \Upsilon_1)\frac{1}{r_{NF} - V_{\min}}(1 - \Pi(t)) - \frac{\Upsilon_2}{V_{\min}}\left(e^{-r_{NF}(T-t)} - \Pi(t)\right) \\
- &\ \frac{bE_0 e^{V_{\max}\tilde{t}_V}}{r_{NF} - 2V_{\min}}\left(e^{V_{\min}\left(t - \tilde{t}_V\right)} - e^{V_{\min}\left(T - \tilde{t}_V\right)}\Pi(t)\right) \\
+ &\ \frac{\Upsilon_3}{r_{NF} - V_{\min}}\left[\left(t + \frac{1}{r_{NF} - V_{\min}}\right) - \left(T + \frac{1}{r_{NF} - V_{\min}}\right)\Pi(t)\right] \\
+ &\ \frac{\Upsilon_4}{r_{NF} - 2V_{\min}}\left(e^{V_{\min}\left(t - \tilde{t}_V\right)} - e^{V_{\min}\left(T - \tilde{t}_V\right)}\Pi(t)\right) \\
+ &\ \frac{\Upsilon_5}{r_{NF} - V_{\min} - \eta}\left(e^{\eta\left(t - \tilde{t}_V\right)} - e^{\eta\left(T\tilde{t}_V\right)}\Pi(t)\right).
\end{aligned} \tag{43}$$

Similarly, for $\lambda_E(t)\,\forall\, t \in [0, \tilde{t}_V]$ we obtain the following expression:

$$\begin{aligned}
\lambda_E(t) &= (a+\Sigma_1)\,\frac{1}{r_{_{NF}}-V_{\max}}\,(1-\Delta(t)) - \frac{bE_0}{r_{_{NF}}-2V_{\max}}\left(e^{V_{\max}t}-e^{V_{\max}\tilde{t}_V}\,\Delta(t)\right)\\
&- \frac{\Sigma_2}{V_{\max}}\left(e^{-r_{_{NF}}\left(\tilde{t}_V-t\right)}-\Delta(t)\right)+\frac{\Sigma_4}{r_{_{NF}}-2V_{\max}}\left(e^{V_{\max}t}-e^{V_{\max}\tilde{t}_V}\Delta(t)\right)\\
&- \frac{\Sigma_3}{r_{_{NF}}-V_{\max}}\left[\left(t+\frac{1}{r_{_{NF}}-V_{\max}}\right)-\left(\tilde{t}_V+\frac{1}{r_{_{NF}}-V_{\max}}\right)\Delta(t)\right]\\
&+ \frac{\Sigma_5}{r_{_{NF}}-V_{\max}-\eta}\left(e^{\eta t}-e^{\eta\tilde{t}_V}\,\Delta(t)\right)+\lambda_E(\tilde{t}_V)\Delta(t),
\end{aligned} \tag{44}$$

where $\lambda_E(\tilde{t}_V)$ in (44) can be obtained from (43) and $\Delta(t) = e^{(r_{_{NF}}-V_{\max})\left(t-\tilde{t}_V\right)}$.

With the two equations for $\lambda_E(t)$ (before and after the switch) the switching time can be obtained. The switching time (provided it is unique) has to satisfy the following first-order condition:

$$\begin{aligned}
\lambda_E(t)E(t) > 0 \quad \forall t \in [0, \tilde{t}_V),\\
\lambda_E(t)E(t) < 0 \quad \forall t \in (\tilde{t}_V, T].
\end{aligned}$$

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